

Short Run SPC Part Four

Standardized Charts for Subgrouped Data

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Lean production is built on the explicit assumption that each step is operated *predictably* and *capably*. Predictable operation can only be achieved and maintained by using process behavior charts. But short production runs make it hard to see how to use process behavior charts. Parts One and Two illustrated how to chart individual values. This paper will illustrate the *Zed-Bar* chart for use with subgrouped data.

SUBGROUPED DATA

When we subgroup values together we are making an explicit assumption that the values within each subgroup were collected under essentially the same conditions. This immediately implies two things: first, that no useful information is lost by averaging the values together, and second, that the differences within the subgroups represent the inherent routine process variation. Hence, subgrouping always requires a judgment call based on the context. It is the internal homogeneity of the subgroups that makes ordinary average and range charts work, and this also holds true for *Zed-Bar* charts.

When several consecutive parts, or nearly consecutive parts, are collected at specified intervals of time, then subgrouping makes sense. However, when the data consist of single values spaced out over time, subgrouping may, or may not, make sense depending on the context.

With short production runs we are likely to have limited amounts of data for each product. If a short production run gives us four values for Product A, we can either plot these four points on a zed chart, or we can combine them and plot one average on a *Zed-Bar* chart. With a zed chart each of the four points can be used for feedback by the operator. With a *Zed-Bar* chart the operator ends up with a report-card value for Product A after the short run is over.

So, unless the context provides a good reason for subgrouping, I recommend plotting individual values with short-run data. However, when you have data which logically belongs in subgroups, and when those subgroups are all the same size, the *Zed-Bar* chart will let you combine subgroups for different products on the same chart.

UNIT 15

Three products are produced in short runs on Unit 15. Samples of five pieces are collected at the start of each run and periodically thereafter throughout the run. Since the runs are of different lengths they usually end up with two, three, or four subgroups for each product run.

Product A was produced in Runs 1, 3, 4, and 6. Product B was produced in Runs 2 and 7.

And Product C was produced in Runs 5 and 8. As always we begin with the product specific charts. In this case they are average and range charts with subgroup size 5. The data are included with each chart.

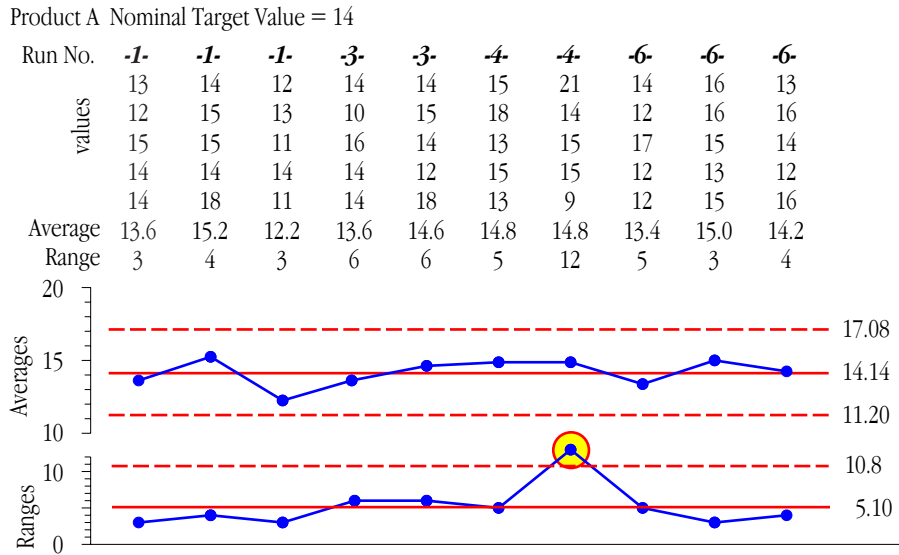


Figure 1: Average and Range Chart for Product A

Product A has a target value of 14, a grand average of 14.14, and an average range of 5.10. Except for the second subgroup of Run 4, they seem to be doing a fairly good job of operating on target.

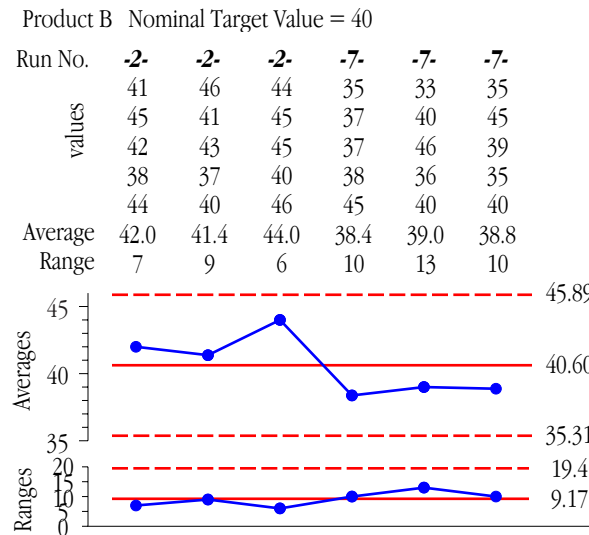


Figure 2: Average and Range Chart for Product B

Product B has a target value of 40, a grand average of 40.60, and an average range of 9.167. While we can see a difference between the Run 2 and Run 7, we do not have quite enough data to

say that this is a detectable difference.

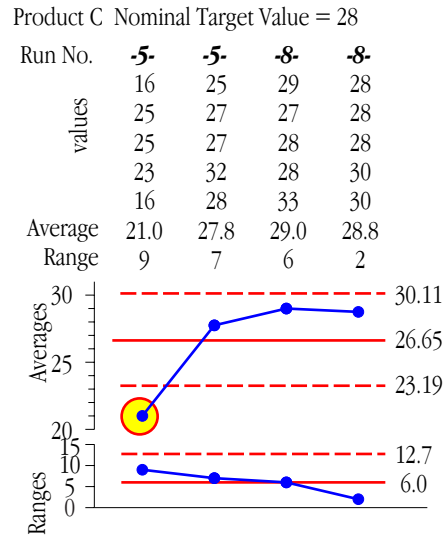


Figure 3: Average and Range Chart for Product C

Product C has a target value of 28, a grand average of 26.65, and an average range of 6.00. Except for the first subgroup of Run 5 they seem to have been operating in the neighborhood of 28.

These product-specific average and range charts set the stage for creating a standardized average and range chart that will allow real-time tracking of the production process following the baseline period.

(Those who may have access to my book *Short Run SPC* should note that the notation used below is different from the notation in chapter six of that book.)

ZED-BAR CHARTS

There are several different, but fully equivalent, versions of the standardized average and range chart. We begin with the simplest version, the *Zed-Bar* and *W* chart. This version of the standardized average and range chart transforms the subgroup averages and subgroup ranges using:

$$Zed-Bar = \frac{Subgroup\ Average - Nominal}{Average\ Range}$$

$$W = \frac{Subgroup\ Range}{Average\ Range}$$

Where both the nominal value and the average range are specific to the product being measured in that subgroup. As before, the nominal value for a product may be either a target value or the baseline grand average value for that product.

The limits for subgroup averages and subgroup ranges are commonly expressed as:

$$\text{Limits for } \bar{X}\text{-bar} = \text{Grand Average} \pm A_2 \text{ Average Range}$$

$$\text{Upper Limit for Ranges} = D_4 \text{ Average Range}$$

$$\text{Central Line for Ranges} = \text{Average Range}$$

$$\text{Lower Limit for Ranges} = D_3 \text{ Average Range}$$

(The scaling factors above may be found in Figure 11.) When we subtract off the grand average (or a target value) we are effectively centering the limits for the subgroup averages on zero. When we divide by the average range we are changing the scale of the limits. Thus, the limits for the *Zed-Bar* and *W* chart become:

$$\text{Limits for } \bar{Z}\text{-Bar} = \text{Zero} \pm A_2 = \text{Zero} \pm \frac{3.00}{d_2 \sqrt{n}}$$

$$\text{Upper Limit for } W = D_4 = 1.00 + \frac{3.00 d_3}{d_2}$$

$$\text{Central Line for } W = 1.00$$

$$\text{Lower Limit for } W = D_3 = 1.00 - \frac{3.00 d_3}{d_2}$$

When the subgroup size is smaller than seven the value for D_3 is negative and there will be no lower limit for the *W* values.

We will use the data from Figures 1, 2, and 3 to illustrate this standardized average and range chart. In this example we will use the target values as the nominals.

ZED-BAR CHART FOR UNIT 15

For the Product A subgroups, $Nominal(A) = 14$ and the average range is $R\text{-bar}(A) = 5.10$, hence the transforms used for Product A averages and ranges are:

$$\bar{Z}\text{-Bar} = \frac{\text{Subgroup Average} - 14}{5.10}$$

and

$$W = \frac{\text{Subgroup Range}}{5.10}$$

For the Product B subgroups, $Nominal(B) = 40$ and $R\text{-bar}(B) = 9.167$, hence the transforms used for Product B averages and ranges are:

$$\bar{Z}\text{-Bar} = \frac{\text{Subgroup Average} - 40}{9.167}$$

and

$$W = \frac{\text{Subgroup Range}}{9.167}$$

And for Product C subgroups, $Nominal(C) = 28$ and $R\text{-bar}(C) = 6.00$, hence the transforms used for Product C averages and ranges are:

$$\bar{Z}\text{-Bar} = \frac{\text{Subgroup Average} - 28}{6.00}$$

and

$$W = \frac{\text{Subgroup Range}}{6.00}$$

When the subgroup averages and ranges from Figures 1, 2, and 3 are placed in production run order and transformed as above, we get our *Zed-Bar* and *W* chart in Figure 4. With subgroups of size $n = 5$ the *Zed-Bar* limits are $\pm A_2 = \pm 0.577$. The *W* chart has a central line of 1.00, an upper limit of $D_4 = 2.114$, and no lower limit.

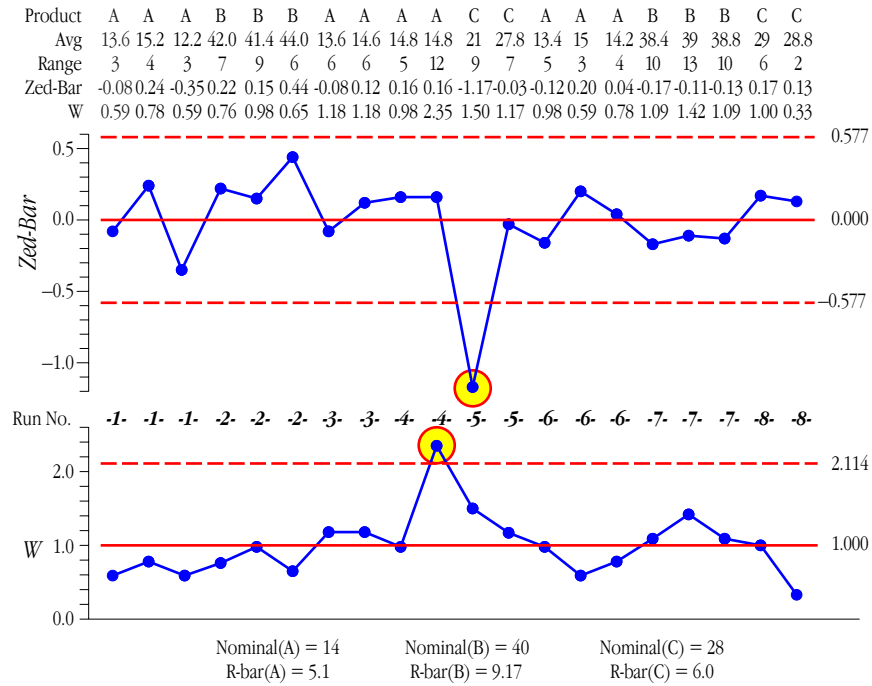


Figure 4: *Zed-Bar* and *W* Chart for Unit 15.

The two points outside the limits in Figure 4 may also be seen in Figures 1 and 3. Here we see that they occurred on successive runs which raises the question of whether these two signals might be related in some way. Except for these two points, the rest of the subgroups Unit 15 seem to have been reasonably close to their target values.

As future production runs of these three products occur the transformed subgroup averages and ranges may be placed on the extension of Figure 4. In this way the process may be visualized in real time while it is used to make different products.

In this version of the standardized average and range chart we used the average range to “standardize” the values plotted, and the limits are given by the standard scaling factors for average and range charts. Thus, a simple division transforms the original averages and ranges into values that allow different products to be placed on the same chart.

VERSION TWO: *ZED-BAR** CHARTS

In the second version of the standardized average and range chart we use a different denominator in our transformations. Instead of the *average range* we use $\text{Sigma}(X)$.

$\text{Sigma}(X)$ is a generic notation for any within-subgroup estimator of the standard deviation for the original values. In the context of an average and range chart $\text{Sigma}(X)$ will generally refer to:

$$\text{Sigma}(X) = \frac{\text{Average Range}}{d_2}$$

Where d_2 is the bias correction factor for subgroups of size n (see Figure 11). Thus:

$$\begin{aligned} \text{Standardized, Centered Subgroup Average} &= \frac{\text{Subgroup Average} - \text{Nominal}}{\text{Sigma}(X)} \\ &= \frac{\text{Subgroup Average} - \text{Nominal}}{\text{Average Range} / d_2} \\ &= [\text{Zed-Bar}] * [d_2] = \text{Zed-Bar}^* \end{aligned}$$

$$\begin{aligned} \text{Standardized Subgroup Range} &= \frac{\text{Subgroup Range}}{\text{Sigma}(X)} \\ &= \frac{\text{Subgroup Range}}{\text{Average Range} / d_2} \\ &= [W] * [d_2] = W^* \end{aligned}$$

Since we can find these standardized values by multiplying the earlier *Zed-Bar* and *W* values by the bias correction factor d_2 , we denote these version two standardized values by *Zed-Bar*^{*} and *W*^{*}.

This also shows us how to find the limits for *Zed-Bar*^{*} and *W*^{*}. We multiply the limits for version one by d_2 . Hence:

$$\text{Limits for Zed-Bar}^* = \text{Zero} \pm d_2 A_2 = \text{Zero} \pm \frac{3.00}{\sqrt{n}}$$

$$\text{Upper Limit for } W^* = d_2 D_4 = d_2 + 3.00 d_3$$

$$\text{Central Line for } W^* = d_2 1.0 = d_2$$

$$\text{Lower Limit for } W^* = d_2 D_3 = d_2 - 3.00 d_3$$

Where d_2 and d_3 are the standard bias correction factors for ranges from subgroups of size n .

For Unit 15 the subgroup size was $n = 5$, so $d_2 = 2.326$, and $d_3 = 0.8641$. Thus, the Limits for the *Zed-Bar*^{*} chart become $2.326 (\pm 0.577) = \pm 1.34$, and the *W*^{*} chart has a central line of 2.326 and an upper limit of 4.918. When the original subgroup averages and ranges for each of the products are transformed with the *Zed-Bar*^{*} and *W*^{*} formulas we get Figure 5.

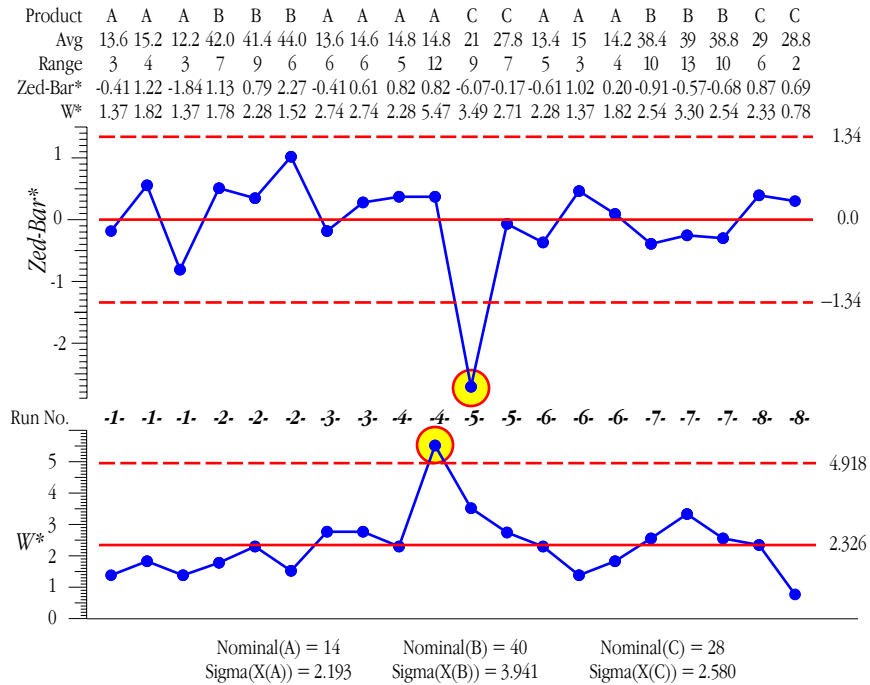


Figure 5: Zed-Bar* and W* Chart for Unit 15.

The *only* difference between Figure 4 and Figure 5 is that they have different vertical scales. While the values plotted are numerically different, the limits change by the same amount, and point by point the story remains the same.

VERSION THREE: ZED-BAR** CHARTS

A third version of the standardized average chart uses yet another denominator for standardizing the subgroup averages. Here we use a within-subgroup estimate of the standard deviation for the subgroup averages, which we shall denote as *Sigma(X-bar)*. When working with average and range charts this estimate is generally taken to be:

$$Sigma(X\text{-bar}) = \frac{Sigma(X)}{\sqrt{n}} = \frac{Average\ Range}{d_2\sqrt{n}}$$

When we standardize the subgroup averages using *Sigma(X-bar)* we end up with the *Zed-Bar*** values.

$$\begin{aligned}
 Zed\text{-Bar}^{**} &= \frac{Subgroup\ Average - Nominal}{Sigma(X\text{-bar})} \\
 &= \frac{Subgroup\ Average - Nominal}{Sigma(X) / \sqrt{n}} = [\sqrt{n}] * [Zed\text{-Bar}^*] \\
 &= [Zed\text{-Bar}] * [\sqrt{n}] * [d_2]
 \end{aligned}$$

Thus, *Zed-Bar*** is simply the original *Zed-Bar* value multiplied by *both* a bias correction factor and the square root of the subgroup size. Hence we label it as *Zed-Bar***.

The limits for *Zed-Bar*** become:

$$\text{Limits for } Zed\text{-Bar}^{**} = \text{Zero} \pm \{ [A_2] * [d_2] * [\sqrt{n}] \} = \text{Zero} \pm 3.00$$

These limits are completely independent of the subgroup size since the subgroup size is taken into account in the transformation of the subgroup averages.

There is no point in dividing the subgroup ranges by *Sigma(X-bar)* because *Sigma(X)* fully standardizes the subgroup ranges. Instead, we commonly use *W** (the standardized ranges from Version Two) along with *Zed-Bar***.

For Unit 15 the limits for the *Zed-Bar*** chart become ± 3.00 , and the *W** chart remains the same as in Figure 5, with a central line of 2.326 and an upper limit of 4.918. When the original subgroup averages and ranges are transformed with the *Zed-Bar*** and *W** formulas we get Figure 6.

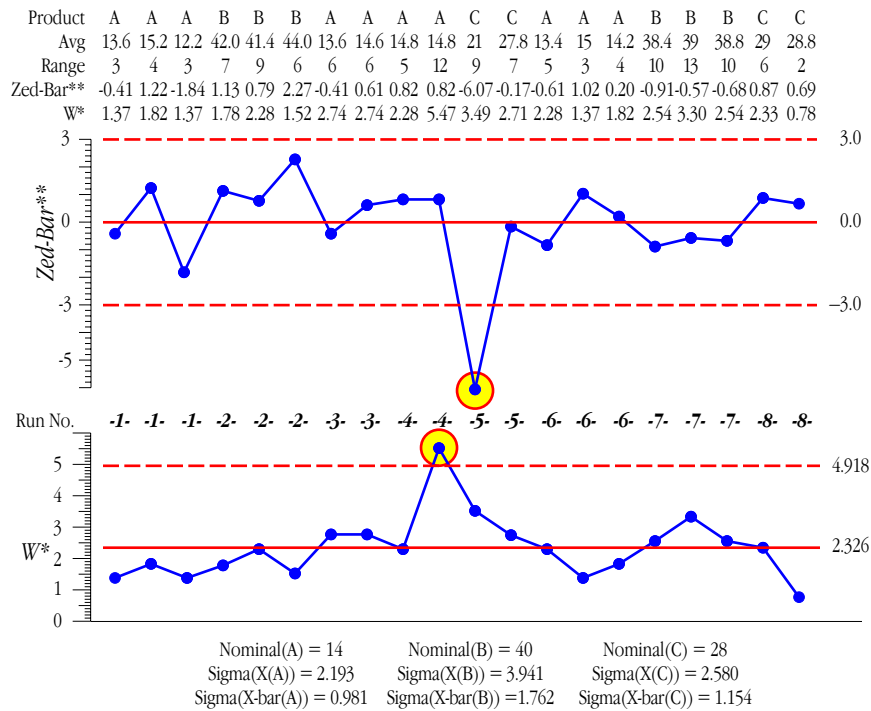


Figure 6: *Zed-Bar*** and *W** Chart for Unit 15.

While the numerical values that are plotted in Figures 4, 5, and 6 change, the limits change with them so that the picture remains the same. The three versions simply use different scales. Given this equivalence, the simplicity of the first version is hard to resist (where we divided everything by the appropriate average range, and used the usual scaling factors as limits). Figures 5 and 6 merely add levels of complexity that serve no real purpose.

WHEN THE NOMINAL IS THE GRAND AVERAGE

When creating a standardized chart the user gets to pick the nominal value. When the nominal values are the target values the standardized chart tracks how closely the process is

operated on-target. When the nominal values are the baseline grand average for each product, the standardized chart will track how well the future production runs match up with the baseline runs.

For Unit 15, Product A has a grand average of 14.14 and a target value of 14. Regardless of which value we use as the nominal, we will get pretty much the same standardized values for Product A.

Product B has a grand average of 40.60 and a target value of 40. Subtracting off a larger value will shift the standardized average values for Product B down slightly compared with Figure 4.

Product C has a grand average of 26.65 and a target value of 28. Here when we use the grand average as the nominal we will get standardized average values that are shifted up relative to Figure 4.

The *Zed-Bar* and *W* chart based on the grand averages as nominals is shown in Figure 7. The white dots show the *Zed-Bar* values from Figure 4 for comparison.

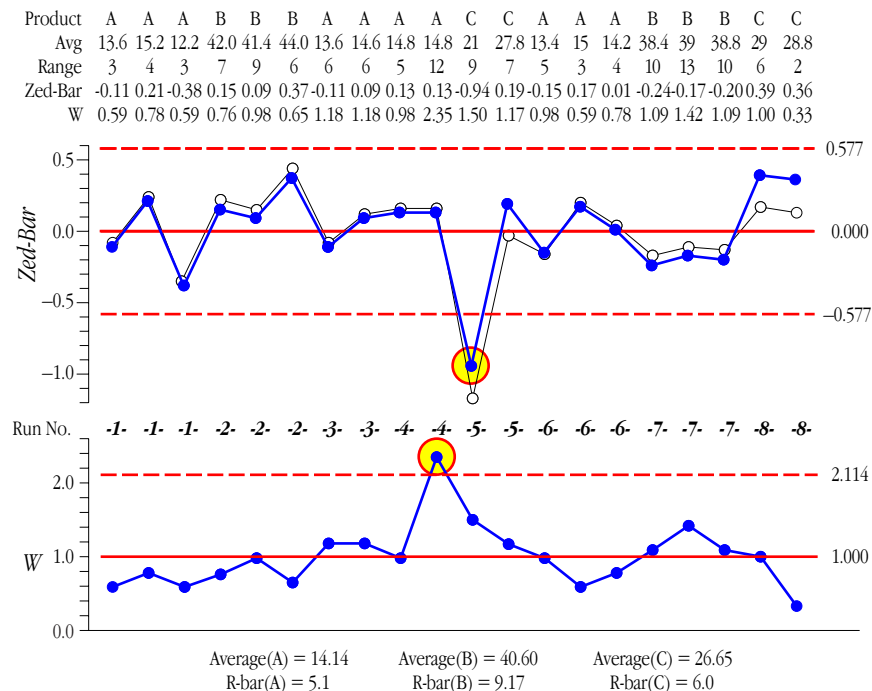


Figure 7: *Zed-Bar* and *W* Chart for Unit 15 with Nominals = Grand Averages.

The choice between a target value and a grand average for the nominal value should be based upon the context for the process being tracked and how you want to use the chart. Here both charts tell the same story because they were (mostly) doing a reasonable job of operating near the target values.

These three versions of a standardized average and range chart are summarized in Figure 8.

Label	Transformation	Central Line	Three-Sigma Limits
1. Zed-Bar	$\frac{\text{Average} - \text{Nominal}}{\text{Average Range}}$	Zero	$\pm A_2 \left(= \frac{\pm 3.00}{d_2 \sqrt{n}} \right)$
W	$\frac{\text{Subgroup Range}}{\text{Average Range}}$	1.00	D_3 and D_4
2. Zed-Bar*	$\frac{\text{Average} - \text{Nominal}}{\text{Average Range}} [d_2]$	Zero	$\frac{\pm 3.00}{\sqrt{n}}$
W*	$\frac{\text{Subgroup Range}}{\text{Average Range}} [d_2]$	d_2	$d_2 D_3$ and $d_2 D_4$
3. Zed-Bar**	$\frac{\text{Average} - \text{Nominal}}{\text{Average Range}} [d_2 \sqrt{n}]$	Zero	± 3.00
W*	$\frac{\text{Subgroup Range}}{\text{Average Range}} [d_2]$	d_2	$d_2 \pm 3 d_3$

Figure 8: Three Versions of the Standardized Average and Range Chart.

VERSION FOUR: STANDARDIZED AVERAGE AND STANDARD DEVIATION CHARTS

In the same way that we standardized the average and range chart we can define Zed-Bar and S charts based on average and standard deviation charts. The Zed-Bar and S chart transforms the subgroup averages and subgroup standard deviation statistics using:

$$\text{Zed-Bar} = \frac{\text{Subgroup Average} - \text{Nominal}}{\text{Average Standard Deviation}}$$

$$S = \frac{\text{Subgroup Standard Deviation}}{\text{Average Standard Deviation}}$$

The limits for the Zed-Bar and S chart are:

$$\text{Limits for Zed-Bar} = \text{Zero} \pm A_3 = \text{Zero} \pm \frac{3.00}{c_4 \sqrt{n}}$$

$$\text{Upper Limit for S} = B_4 = 1 + \frac{3}{c_4} \sqrt{1 - (c_4)^2}$$

$$\text{Central Line for S} = 1.0$$

$$\text{Lower Limit for S} = B_3 = 1 - \frac{3}{c_4} \sqrt{1 - (c_4)^2}$$

Where c_4 is the bias correction factor for subgroup standard deviation statistics from subgroups of size n .

For Product A subgroups, $\text{Nominal}(A) = 14$ and $s\text{-bar}(A) = 2.00$, hence the transforms used for Product A averages and standard deviations would be:

$$\text{Zed-Bar} = \frac{\text{Subgroup Average} - 14}{2.00}$$

and

$$S = \frac{\text{Subgroup Std. Dev.}}{2.00}$$

For Product B subgroups, $Nominal(B) = 40$ and $s\text{-bar}(B) = 3.56$, hence the transforms used for Product B averages and standard deviations would be:

$$Zed\text{-Bar} = \frac{\text{Subgroup Average} - 40}{3.56}$$

and

$$S = \frac{\text{Subgroup Std. Dev.}}{3.56}$$

And for Product C subgroups, $Nominal(C) = 28$ and $s\text{-bar}(C) = 2.67$, hence the transforms used for Product C averages and standard deviations would be:

$$Zed\text{-Bar} = \frac{\text{Subgroup Average} - 28}{2.56}$$

and

$$S = \frac{\text{Subgroup Std. Dev.}}{2.56}$$

When the subgroups are placed in production order and transformed as above, the *Zed-Bar* chart will have a central line of zero and limits of $\pm A_3$ where A_3 is the usual scaling factor for an average chart for subgroups of size n . With subgroups of size $n = 5$ the value for A_3 is 1.427.

The *S* chart will have a central line of 1.00 and limits of B_3 and B_4 . For subgroups of size $n = 5$, $B_4 = 2.089$ and there is no lower limit. Figure 8 shows the *Zed-Bar* and *S* chart for Unit 15.

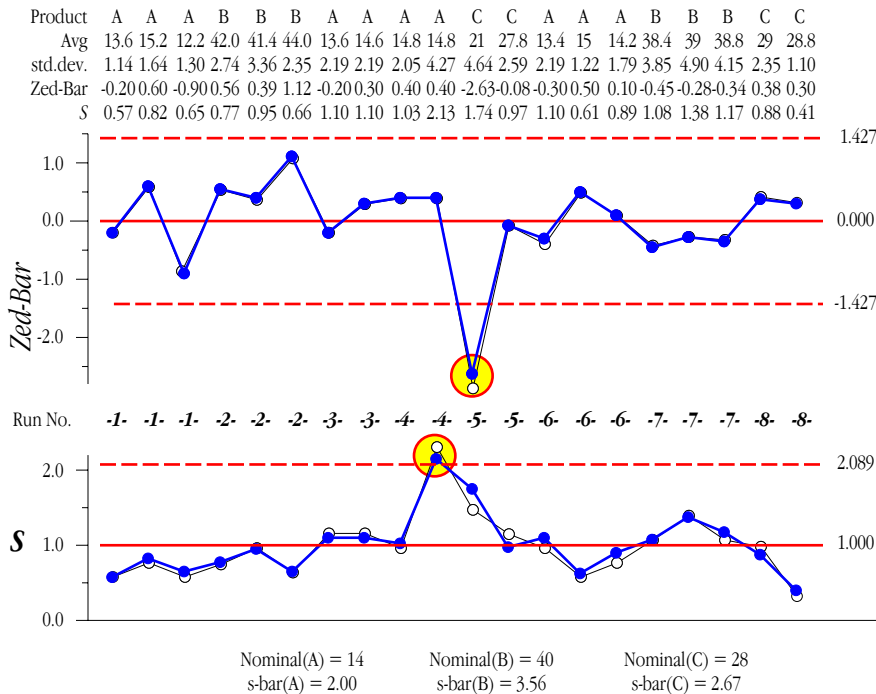


Figure 9: *Zed-Bar* and *S* Chart for Unit 15

The white dots show the equivalent running record from Figure 4 for the sake of comparison.

Figure 8 tells the same story as Figures 4, 5, 6, and 7. This happens because the subgroup ranges and subgroup standard deviations are *always* highly correlated for subgroup sizes of 15 or less. With subgroups of size five the ranges and standard deviation statistics always have a correlation coefficient of 97.5%, hence it makes virtually no difference whether we use an average and range chart or an average and standard deviation chart.

Of course, we can also define additional versions of the standardized average and standard deviation chart in the manner done before. When we do this we end up with the limits shown in Figure 10 for these various versions.

Label	Transformation	Central Line	Three-Sigma Limits
4. Zed-Bar	$\frac{\text{Average} - \text{Nominal}}{\text{Avg. Std. Dev.}}$	zero	$\pm A_3 \left(= \frac{\pm 3.00}{c_4 \sqrt{n}} \right)$
S	$\frac{\text{Subgroup Std.Dev.}}{\text{Avg. Std. Dev.}}$	1.00	B_3 and B_4
5. Zed-Bar*	$\frac{\text{Average} - \text{Nominal}}{\text{Avg. Std. Dev.}} [c_4]$	Zero	$\frac{\pm 3.00}{\sqrt{n}}$
S*	$\frac{\text{Subgroup Std.Dev.}}{\text{Avg. Std. Dev.}} [c_4]$	c_4	$c_4 B_3$ and $c_4 B_4$
6. Zed-Bar**	$\frac{\text{Average} - \text{Nominal}}{\text{Avg. Std. Dev.}} [c_4 \sqrt{n}]$	Zero	± 3.00
S*	$\frac{\text{Subgroup Std. Dev.}}{\text{Avg. Std. Dev.}} [c_4]$	c_4	$c_4 \pm 3 \sqrt{1 - (c_4)^2}$

Figure 10: Three Versions of Standardized Average and Standard Deviation Charts

WORKING WITH SOFTWARE

When working with software that attempts to present a standardized average and range chart you might find any one of the six versions above. In practice you can usually determine which transformation has been used by looking at the values for the limits on the chart.

If you are unable to puzzle it out directly, you may evaluate your software by using the data given here. If you do this and do not find the equivalent of Figure 4, 5, 6, 7, or 8, your software may be standardizing the data in some incorrect or inappropriate manner (e.g. as covered in Part Three).

HOW MANY DATA

The quality of a Zed-Bar chart will depend upon how many data are used for each product. When using fewer than 12 observations the statistics used will be soft and uncertain. With 12 to 33 observations the statistics will gel and solidify. Once we have more than 33 values in the baseline for a product our limits will be solid enough that additional data will make little difference in the resulting charts.

<i>n</i>	Average Range					Avg. Std. Dev.			
	<i>A</i> ₂	<i>D</i> ₃	<i>D</i> ₄	<i>d</i> ₂	<i>d</i> ₃	<i>A</i> ₃	<i>B</i> ₃	<i>B</i> ₄	<i>c</i> ₄
2	1.880	–	3.267	1.128	.8525	2.659	–	3.267	.7979
3	1.023	–	2.574	1.693	.8884	1.954	–	2.568	.8862
4	0.729	–	2.282	2.059	.8798	1.628	–	2.266	.9213
5	0.577	–	2.114	2.326	.8641	1.427	–	2.089	.9400
6	0.483	–	2.004	2.534	.8480	1.287	0.030	1.970	.9515
7	0.419	0.076	1.924	2.704	.8332	1.182	0.118	1.882	.9594
8	0.373	0.136	1.864	2.847	.8198	1.099	0.118	1.815	.9650
9	0.337	0.184	1.816	2.970	.8078	1.032	0.239	1.761	.9693
10	0.308	0.223	1.777	3.078	.7971	0.975	0.284	1.716	.9727

Figure 11: Scaling Factors and Bias Correction Factors for *Zed-Bar* Charts

RECOMMENDATIONS

Whenever the data are limited or when the data are collected periodically you should always begin with charts for individual values. If the process tends to change slowly relative to the sample frequency, then such data may be subgrouped in appropriate ways with little loss of information.

When the data are obtained in clusters, the subgrouping of these data and the use of *Zed-Bar* charts may be appropriate. If the chart is to be used by those running the process one should be careful to avoid subgrouping too much data together. With large subgroup sizes, or a low subgroup frequency, short production runs can result in charts that are nothing but ancient history. With short production runs it is often better to measure fewer parts more frequently and to use charts for individual values, than it is to subgroup larger amounts of data that are obtained less often.

