Power Functions for Process Behavior Charts

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Every data set contains noise (random, meaningless variation). Some data sets contain signals (nonrandom, meaningful variation). Virtually every statistical procedure is aimed at detecting the presence of signals in spite of the noise. Yet different statistical procedures react to the presence of a signal in different ways. How might we compare different procedures?

One simple way is to analyze a single data set using different techniques and compare the results. While this type of comparison is straightforward and easy to perform, it has the drawback of not being completely general. It is a narrow, empirical comparison which may be affected by peculiar characteristics of the data set being analyzed. Thus, while such simple comparisons are useful as preliminary evaluations, a more comprehensive approach is needed for a rigorous comparison.

Mathematical theory will allow us to make comparisons between procedures. These comparisons will use probability models rather than actual data, and therefore are completely general and global. Unfortunately, such comparisons will inevitably have to be based upon some set of assumptions, and these assumptions are always either unverifiable, or unrealistic, or both.

Thus, empirical comparisons are not general, while theoretical comparisons are more general but unrealistic. Given this conundrum, we shall choose to use theoretical comparisons for the sake of their generality, but we shall also have to be aware that only gross differences in theoretical performance are likely to transfer over into the real world.

In other words, we are going to adopt the probability approach here as a matter of convenience—not as a guide to practice, and not as a means of justifying process behavior charts, *but as an expedient tool for examining how the process behavior chart functions with different detection rules*.

Since power functions are theoretical, small differences in power will be irrelevant in practice. However, experience has shown that large differences in theoretical power will usually translate into detectable differences in sensitivity in practice—the procedure with the greatest theoretical power will generally be the more sensitive in practice. It is this rough correlation with practice which justifies the use of power functions.

The Power of Charts for Location

Average Charts and Charts for Individual Values may both be considered as charts which track the "process location." The only distinction made between these two types of charts in the following material is by reference to the subgroup size. When $n = 1$, the chart described will be the chart for individual values, otherwise it will be a chart for subgroup averages.

For process behavior charts that track process location, the "power" of the chart is defined to be the (theoretical) probability that the chart will detect a particular shift in the process location. Of course, as the size of the shift changes so will the power for that process behavior chart. When the different power values are plotted against the different shifts in process location the result is a power function for a given process behavior chart.

As is suggested by the word "theoretical," the values of the power function are traditionally found by making certain simplifying assumptions. Among these are the following:

(1) the measurements do not display any discreteness;
(2) the measurements are independently and normally distributed;
(3) the three-sigma limits are known without error; and
(4) the changes in the process location can be represented by a step function.

While these assumptions make the theoretical problem easier to work, they are all, to a greater or a lesser degree, unrealistic in practice.

In general, a power function will depend upon three things:

(1) the size of the signal to be detected;
(2) the amount of data available to detect the signal; and
(3) the procedure used to analyze these data.

For process behavior charts for location, the size of the signal is taken to be the size of the shift in the process location. The amount of data available is determined by the subgroup size and the number of subgroups following the shift. And the procedure is determined by the detection rules used to detect the shift. For the purposes of this paper we shall begin with the Western Electric Zone Tests.

Detection Rule One: A point outside a three-sigma limit is taken as a signal of exceptional variation that is attributable to a dominant assignable cause.

Detection Rule Two: Two out of three successive values on the same side of the central line and more than two standard deviations away from the central line are taken as a signal of a moderate but sustained shift in the process.

Detection Rule Three: Four out of five successive values on the same side of the central line and more than one standard deviation away from the central line are taken as a signal of a small but sustained shift in the process.

Detection Rule Four: Eight successive values on the same side of the central line are taken as a signal of a small but sustained shift.

The presence of any one of these four conditions implies that enough evidence has been accumulated to make you reasonably confident that there has been a change in the process.

The size of a shift in the process location may be expressed in different ways. A shift in location may be expressed in measurement units, or it may be expressed as a multiple, Δ, of the standard deviation of the distribution of X, $SD(X)$, and at other times may be expressed as a multiple, δ, of the standard deviation of the subgroup average. This last standard deviation is traditionally called the "standard error" and will be denoted by the symbol "s.e." This multiplicity of representations for a shift in location is illustrated in Figure 1. There a shift of 2.7 measurement units is shown. Since $Sigma(X) = 1.80$ for the process approximated by these curves, this shift of 2.7 measurement units is equal to 1.5 standard deviations of X. Since n = 5, this shift of 1.5 standard deviations of X is also equal to 3.35 standard errors.

The use of standard error units allows a dramatic reduction in the size of the tables needed to express the power functions, while the use of $SD(X)$ units is needed to make direct comparisons between different process behavior charts. The relationship between these two different systems is seen in the following:

$$\delta \text{ s.e.} = \delta \; SD(\bar{X}) = \delta \; \frac{SD(X)}{\sqrt{n}} = \frac{\delta}{\sqrt{n}} \; SD(X) = \Delta \; SD(X)$$

So, to convert δ into Δ, divide δ by $\sqrt{n}$, and to convert Δ into δ multiply Δ by $\sqrt{n}$.

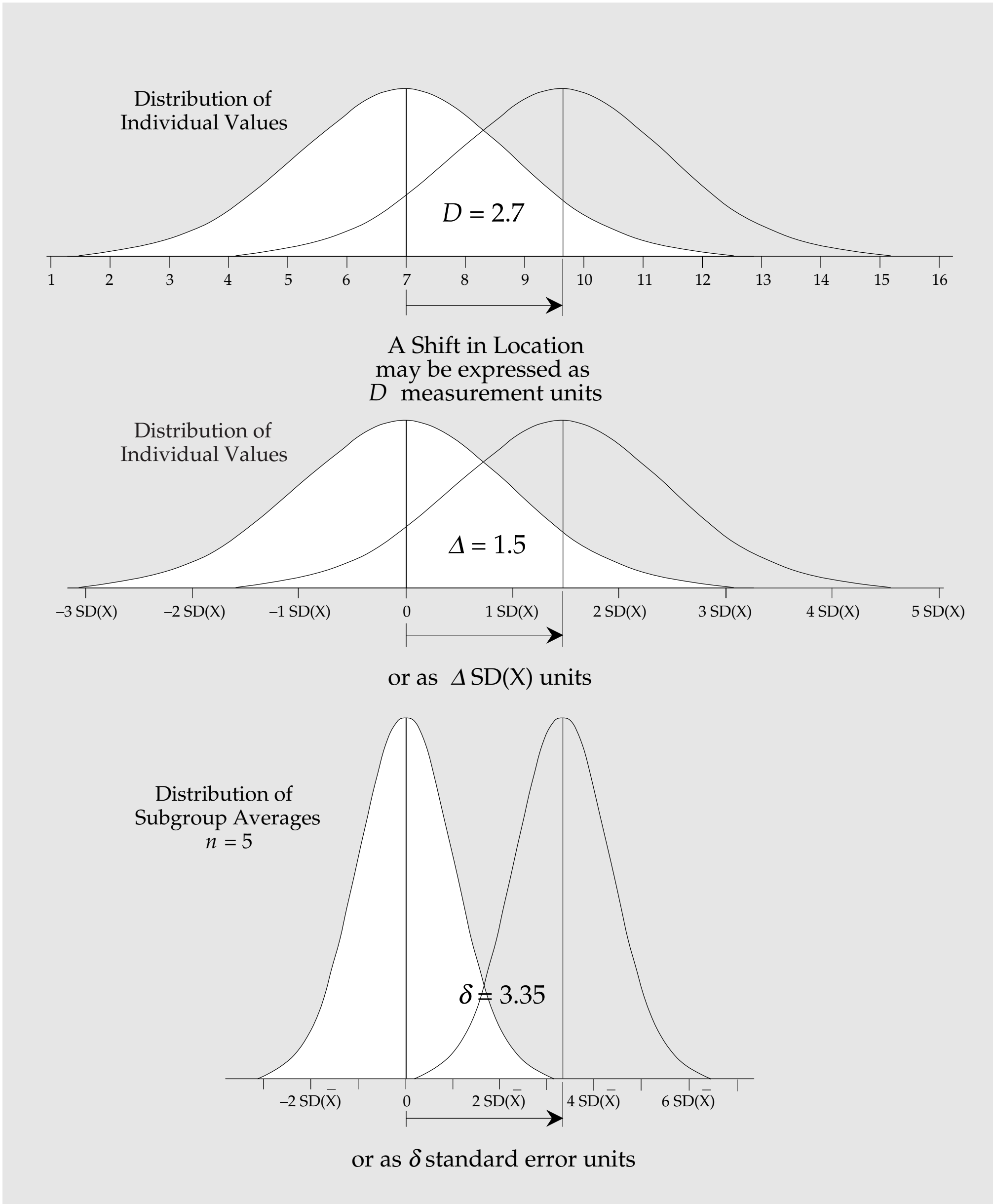

Figure 1: Three Ways to Express a Shift in Location

The power functions plotted in Figures 2, 3, and 4 are based upon the tables of the power function given later. Figure 2 shows the power functions for the use of $k = 1$ subgroup of size n. Of course, with only one subgroup, we can only use Detection Rule One. The separate curves are for different subgroup sizes, beginning with $n = 1$ and going up to $n = 10$.

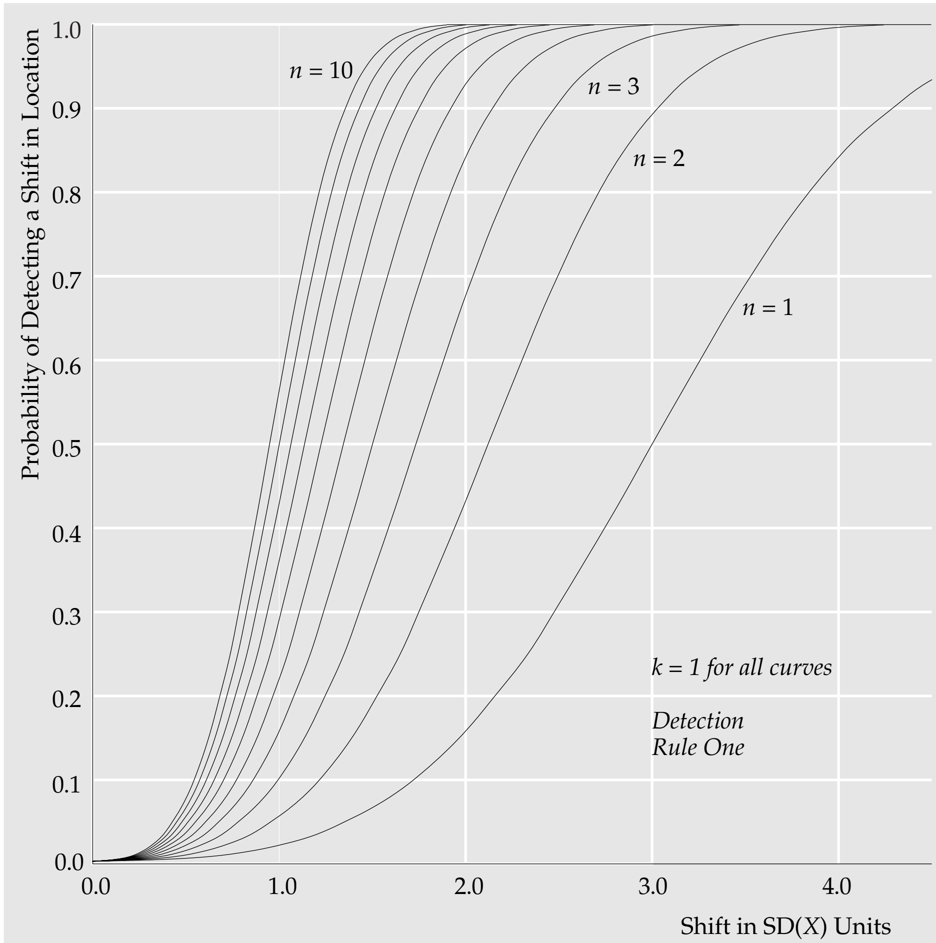

Figure 2: The Effect of Subgroup Size Upon Power

The vertical axis in Figure 2 is the theoretical probability of detecting a shift using the given amount of data. The horizontal axis shows the size of the shift in *SD(X)* units. In this graph, the steeper the curve, and the further to the left, the greater the theoretical power of the procedure. Note that all these curves show a very small "probability" of detecting a "shift of 0.0 units." Such small intercepts are desirable. Thus, when interpreting a power function curve we look for a curve which is low at a shift of 0.0 and which increases rapidly as it moves to the right.

Figure 2 shows an increase in the power of the process behavior chart which corresponds to each increase in the subgroup size. This would suggest that larger subgroup sizes would tend to be preferred. However, further inspection of Figure 2 shows that the curves get closer together as *n* increases. This suggests a certain diminishing return for each incremental increase in the subgroup size. Moreover, the increase in power with increasing *n* shown in Figure 2 will only occur as long as the subgroups are logically homogeneous. A lack of homogeneity within the subgroups will more than offset any apparent advantage of having a larger value for n. Therefore, we must make the subgroup size a *secondary* consideration. The rational homogeneity of the subgroups must still be *primary*.

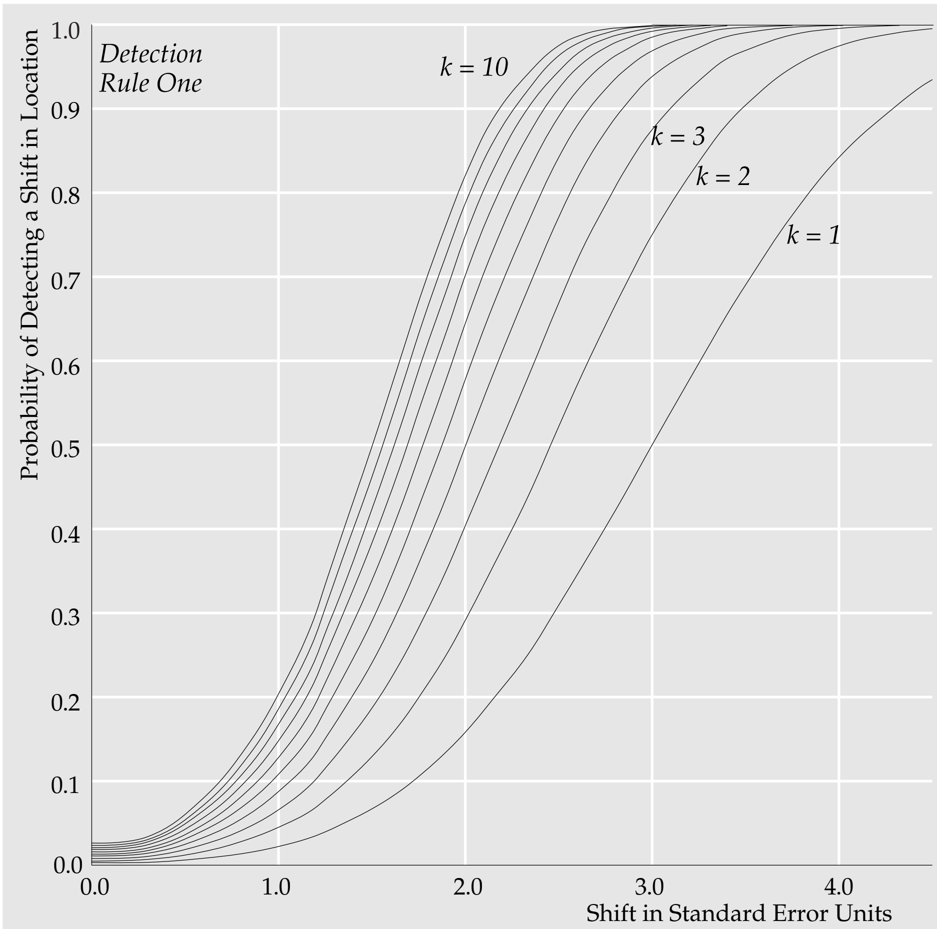

Figure 3: The Effect of Multiple Subgroups Upon Power

Figure 3 shows the effect of increasing the number of subgroups for any fixed subgroup size. The curves in Figure 3 use only Detection Rule One and are plotted as a function of standard error units. Once again, as more data are collected following the shift, the greater the likelihood that the shift will be detected, albeit with diminishing returns. However, for shifts greater than 4 standard errors, even one subgroup is very likely to detect the shift (regardless of the subgroup size). The real increase in sensitivity comes for shifts in location which are between 2 s.e. units and 4 s.e. units in size. In other words, big shifts will be detected immediately, while smaller shifts will be detected eventually. Since the economic impact of larger shifts is likely to be greater than that of smaller shifts, this is perfectly appropriate. Finally, you should notice that the intercepts with the vertical axis in Figure 3 do show a slight increase as k increases. This increase in the probability of a false alarm is inherent in any repeated application of a statistical procedure.

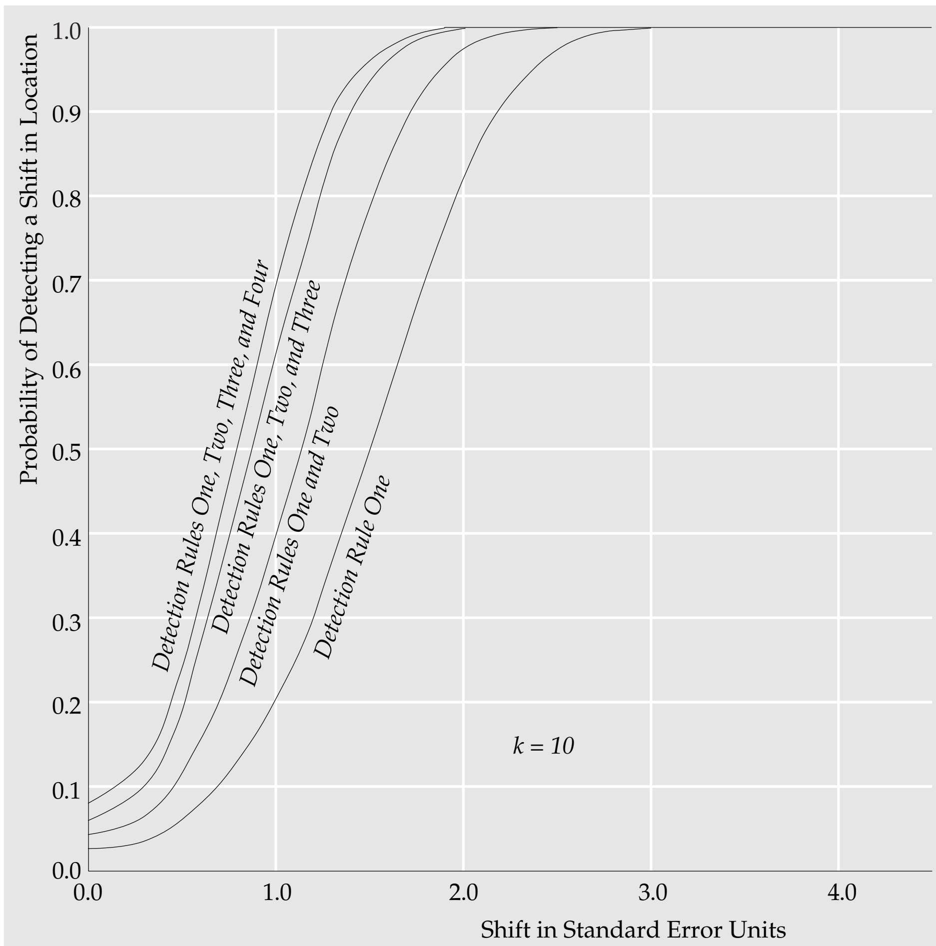

Figure 4: The Effect of Multiple Detection Rules Upon Power

Figure 4 shows the effect of the use of multiple detection rules. Figure 4 was created using $k = 10$ subgroups. For a fixed amount of data, the use of Detection Rules Two, Three, and Four increase the power of the procedure. Once again, there is a diminishing return apparent in the power, but you should note that there is not the same diminishing return for the intercepts with the vertical axis. This increase in the false alarm error rate is the main reason that you should not use too many detection rules. Each additional detection rule will increase the likelihood of false alarms, yet after a few rules are in use, the addition of a new rule will not appreciably increase the power.

As the graphs in Figures 2 and 3 suggest, when dealing with regular process behavior chart data, you may increase the sensitivity of the chart for averages by simply increasing the subgroup size. If this is not feasible due to the nature of the process, then you may increase the sensitivity by collecting subgroups more often. In both cases, the sensitivity increases in proportion to the square root of the amount of data available for detecting the shift. At the same time the cost of collecting the data increases in direct proportion to the amount of data. Thus, there will always be a diminishing return for each additional observation. Increasing the subgroup size makes the individual subgroup more responsive, while increasing the number of subgroups simply offers more opportunities for the subgroup averages to detect the shift.

Unlike the changes above, the use of additional detection rules does not *immediately* increase the sensitivity of the process behavior chart. You must have at least two subgroups to utilize Detection Rule Two. Detection Rule Three requires at least four subgroups, and Detection Rule Four requires at least eight.

In practice, Detection Rule One will usually generate all the signals that most people can realistically investigate. And since investigation of out-of-limit points is the key to the effective use of process behavior charts, you should be careful about introducing too many detection rules. The charts are intended to be a basis for *action*, rather than an unending *nag* which you must ignore in self-defense.

The Formulas for the Power Functions

Assume that a sequence of measurements is produced by some process, and assume that this process has displayed a reasonable degree of predictability in the past. Furthermore, assume that the measurements are independently and normally distributed, that measurements can be made to any desired number of digits, and that the process parameters are known. (Remember, this is done for the sake of a theoretical comparison, not as a prescription of what conditions must be present before you may use a process behavior chart.)

Now consider the process behavior chart for location (either an Individual Values Chart or an Average Chart) as a device for detecting a change in the process mean. The power of this chart for this application will be a function of three things:

(1) the size of the shift,

(2) the number of subgroups or values collected following the shift, and

(3) the way these data are organized into subgroups for the chart.

The formulas for the power functions for charts for location are given in this section. The material in this section comes in part from Donald J. Wheeler, "Detecting a Shift in the Process Average: Tables of the Power Function of X-Bar Charts," *Journal of Quality Technology, v.15*, pp. 155-170, 1983. The more recent material was developed by Rip Stauffer. The tables given here have been recomputed using these additional formulas.

Detection Rule One Alone

When working with Detection Rule One there is only one way to detect a signal—a point has to actually fall outside a three-sigma limit. Given a particular shift in the process location, let the symbol "a" denote the probability that a point is beyond the three-sigma limit on the same side as the shift.

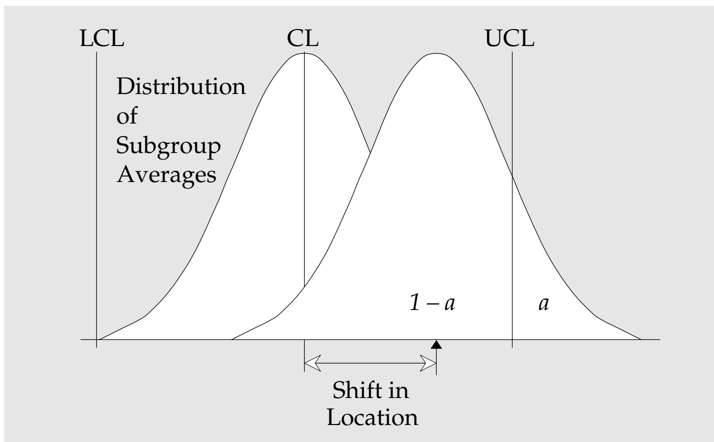

Figure 5: The Probability of Exceeding the Limit for a Given Shift in Location

Figure 5 shows a shift in location. The probability of detecting this shift on the *first* subgroup following the shift is:

$$\textit{Probability of detection on } 1^{st} \textit{ subgroup following shift} = p_1 = a$$

The probability of detecting the shift on the *second* subgroup following the shift is:

$$\textit{Probability of detection on } 2^{nd} \textit{ subgroup following shift} = p_2 = a\,(1\text{-}a)$$

Continuing in this manner, the probability of detecting the shift on exactly the k^{th} subgroup following the shift is:

$$\textit{Probability of detection on } k^{th} \textit{ subgroup following shift} = p_k = a\;(1\text{–}a)^{\,k-1}$$

Therefore, the probability of detecting a shift *within the first* k subgroups following the shift is found to be:

$$\textit{Probability of detecting shift within k subgroups following shift} = \sum_{i=1}^{k} p_i$$

$$= \sum_{i=1}^{k} a\;(1-a)^{\,i-1}$$

$$= 1 - (1-a)^k$$

This result has been known for many years. This formula may be used with the value of *a* obtained from any appropriate probability distribution.

Using the normal distribution, if we express the shift in standard error units as δ, then the probability that the location statistic will exceed the three-sigma limit on the same side as the shift is:

$$\textit{Probability of } Z > (3 - \delta) \quad = a$$

where Z is the standard normal variable.

The shifts shown in Table 1 are expressed in standard error units, δ. The values in the tables are the probabilities of detecting a shift *within k subgroups following the shift*.

Table 1: Power Functions for Charts for Location Using Detection Rule One Alone

shift δ	***k=1***	***k=2***	***k=3***	***k=4***	***k=5***	***k=6***	***k=7***	***k=8***	***k=9***	***k=10***
0	0.003	0.005	0.008	0.011	0.013	0.016	0.019	0.021	0.024	0.027
0.1	0.002	0.004	0.006	0.007	0.009	0.011	0.013	0.015	0.017	0.019
0.2	0.003	0.005	0.008	0.010	0.013	0.015	0.018	0.020	0.023	0.025
0.3	0.003	0.007	0.010	0.014	0.017	0.021	0.024	0.027	0.031	0.034
0.4	0.005	0.009	0.014	0.019	0.023	0.028	0.032	0.037	0.041	0.046
0.5	0.006	0.012	0.019	0.025	0.031	0.037	0.043	0.049	0.055	0.060
0.6	0.008	0.016	0.024	0.032	0.040	0.048	0.056	0.064	0.071	0.079
0.7	0.011	0.021	0.032	0.042	0.052	0.063	0.073	0.083	0.092	0.102
0.8	0.014	0.028	0.041	0.054	0.068	0.081	0.093	0.106	0.118	0.131
0.9	0.018	0.035	0.053	0.070	0.086	0.103	0.119	0.134	0.150	0.165
1.0	0.023	0.045	0.067	0.088	0.109	0.129	0.149	0.168	0.187	0.206
1.1	0.029	0.057	0.084	0.110	0.136	0.160	0.185	0.208	0.231	0.253
1.2	0.036	0.071	0.104	0.136	0.167	0.197	0.226	0.254	0.281	0.306
1.3	0.045	0.087	0.128	0.167	0.204	0.239	0.273	0.306	0.337	0.366
1.4	0.055	0.107	0.156	0.202	0.246	0.287	0.326	0.363	0.398	0.431
1.5	0.067	0.129	0.187	0.242	0.292	0.340	0.384	0.425	0.463	0.499
1.6	0.081	0.155	0.223	0.286	0.344	0.397	0.445	0.490	0.531	0.569
1.7	0.097	0.184	0.263	0.335	0.399	0.457	0.510	0.557	0.600	0.639
1.8	0.115	0.217	0.307	0.387	0.457	0.520	0.575	0.624	0.667	0.705
1.9	0.136	0.253	0.354	0.442	0.518	0.583	0.640	0.689	0.731	0.767
2.0	0.159	0.292	0.404	0.499	0.578	0.645	0.702	0.749	0.789	0.822
2.1	0.184	0.334	0.457	0.557	0.638	0.705	0.759	0.804	0.840	0.869
2.2	0.212	0.379	0.510	0.614	0.696	0.760	0.811	0.851	0.883	0.908
2.3	0.242	0.425	0.564	0.670	0.750	0.810	0.856	0.891	0.917	0.937
2.4	0.274	0.473	0.618	0.723	0.799	0.854	0.894	0.923	0.944	0.959
2.5	0.309	0.522	0.669	0.771	0.842	0.891	0.924	0.948	0.964	0.975
2.6	0.345	0.570	0.718	0.815	0.879	0.921	0.948	0.966	0.978	0.985
2.7	0.382	0.618	0.764	0.854	0.910	0.944	0.966	0.979	0.987	0.992
2.8	0.421	0.664	0.806	0.887	0.935	0.962	0.978	0.987	0.993	0.996
2.9	0.460	0.709	0.843	0.915	0.954	0.975	0.987	0.993	0.996	0.998
3.0	0.500	0.750	0.875	0.937	0.969	0.984	0.992	0.996	0.998	0.999

Table 1: Power Functions for Charts for Location Using Detection Rule One Alone

shift δ	*k=1*	*k=2*	*k=3*	*k=4*	*k=5*	*k=6*	*k=7*	*k=8*	*k=9*	*k=10*
3.1	0.540	0.788	0.903	0.955	0.979	0.991	0.996	0.998	0.999	1.000
3.2	0.579	0.823	0.926	0.969	0.987	0.994	0.998	0.999	1.000	
3.3	0.618	0.854	0.944	0.979	0.992	0.997	0.999	1.000		
3.4	0.655	0.881	0.959	0.986	0.995	0.998	0.999			
3.5	0.691	0.905	0.971	0.991	0.997	0.999	1.000			
3.6	0.726	0.925	0.979	0.994	0.998	1.000				
3.7	0.758	0.941	0.986	0.997	0.999					
3.8	0.788	0.955	0.990	0.998	1.000					
3.9	0.816	0.966	0.994	0.999						
4.0	0.841	0.975	0.996	0.999						
4.1	0.864	0.982	0.998	1.000						
4.2	0.885	0.987	0.998							
4.3	0.903	0.991	0.999							
4.4	0.919	0.993	0.999							
4.5	0.933	0.996	1.000							
4.6	0.945	0.997								
4.7	0.955	0.998								
4.8	0.964	0.999								
4.9	0.971	0.999								
5.0	0.977	0.999								
5.1	0.982	1.000								
5.2	0.986									
5.3	0.989									
5.4	0.992									
5.5	0.994									
5.6	0.995									
5.7	0.997									
5.8	0.997									
5.9	0.998									
6.0	0.999									

By dividing the shift in standard error units shown in the table above by the square root of the subgroup size, the values in the first column of Table 1 can be used to create the curves shown in Figure 2.

The ten curves shown in Figure 3 can be obtained by plotting each of the columns in Table 1.

In Figure 4 the curve labeled "Detection Rule One" is found by plotting the last column of values in Table 1 above.

To account for false alarms on both sides when $\delta = 0$ the formula probabilities are doubled in the first rows of Tables 1 through 6.

Detection Rules One and Two

When Detection Rules One and Two are used together the computations become more complex. The probability of detecting a shift on the first subgroup following the shift is still the same as above:

$$\textit{Probability of detection on } 1^{st} \textit{ subgroup following shift } = p_1 = a$$

Detecting a shift on the second subgroup following the shift can happen in two ways: (1) both subgroup averages fall between one of the two-sigma lines and the adjacent three-sigma limit, or (2) the first subgroup value does not fall outside the three-sigma limit but the second subgroup value does. Figure 6 shows the labels for the different probabilities associated with the use of Detection Rules One and Two.

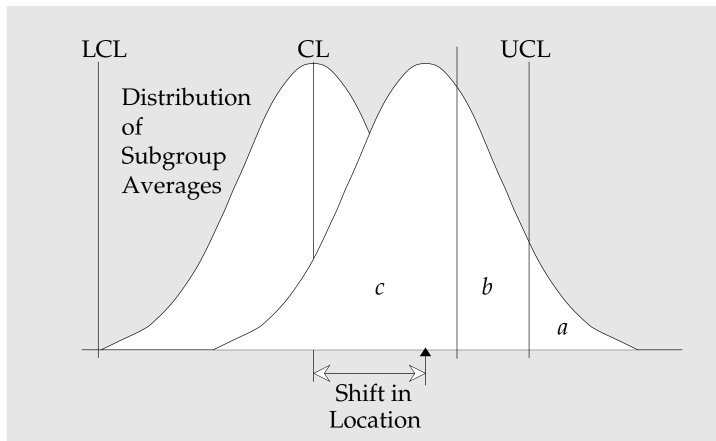

Figure 6: Probabilities for Detection Rules One and Two

The probability that two successive subgroup values will fall between the two-sigma line and the adjacent three-sigma limit is b^2. The probability that the first subgroup value is inside the limits and the second value is outside the three-sigma limit is $a(b+c)$. Thus:

$$p_2 = b^2 + a(b+c)$$

In a similar vein the formulas for detecting a shift on the k^{th} subgroup following the shift are found to be:

$$p_3 = 2\,abc + 2\,b^2c + ac^2$$

$$p_4 = 3\,abc^2 + 2\,b^2c^2 + ac^3$$

$$p_5 = ab^2c^2 + b^3c^2 + 4\,abc^3 + 2\,b^2c^3 + ac^4$$

$$p_6 = 3\,ab^2c^3 + 3\,b^3c^3 + 5\,abc^4 + 2\,b^2c^4 + ac^5$$

$$p_7 = 6\,ab^2c^4 + 5\,b^3c^4 + 6\,abc^5 + 2\,b^2c^5 + ac^6$$

$$p_8 = ab^3c^4 + b^4c^4 + 10\,ab^2c^5 + 7\,b^3c^5 + 7\,abc^6 + 2\,b^2c^6 + ac^7$$

$$p_9 = 4\,ab^3c^5 + 4\,b^4c^5 + 15\,ab^2c^6 + 9\,b^3c^6 + 8\,abc^7 + 2\,b^2c^7 + ac^8$$

$$p_{10} = 11\,ab^3c^6 + 10\,b^4c^6 + 21\,ab^2c^7 + 10\,b^3c^7 + 8\,abc^8 + 2\,b^2c^8 + ac^9$$

The formulas above completely enumerate the 191 ways that Detection Rules One and Two can detect a shift within 10 subgroups following that shift. These formulas are perfectly general—they will work with values of *a*, *b*, and *c* obtained from any appropriate probability distribution.

By summing these probabilities we can obtain the probability of detecting a shift within *k subgroups* following the shift (for $k = 1$ to 10) using Detection Rules One and Two together. Using the normal distribution, if we express the shift in standard error units as δ, then the probabilities needed for the formulas above will be:

$$\textit{Probability of } Z > (3 - \delta) = a$$
$$\textit{Probability of } (2 - \delta) < Z < (3 - \delta) = b$$
$$c = 1 - a - b$$

where Z is the standard normal variable. As in Table 1, the shifts shown in Table 2 are expressed in standard error units, δ. The values shown are the probabilities of detecting a shift *within k subgroups following the shift*.

Table 2: Power Functions for Charts for Location: Detection Rules One and Two

shift δ	***k=1***	***k=2***	***k=3***	***k=4***	***k=5***	***k=6***	***k=7***	***k=8***	***k=9***	***k=10***
0	0.003	0.009	0.013	0.018	0.022	0.027	0.031	0.036	0.038	0.042
0.1	0.002	0.004	0.008	0.011	0.014	0.017	0.020	0.024	0.027	0.030
0.2	0.003	0.006	0.011	0.016	0.020	0.025	0.029	0.034	0.038	0.042
0.3	0.003	0.009	0.015	0.022	0.028	0.035	0.041	0.047	0.053	0.059
0.4	0.005	0.012	0.021	0.030	0.039	0.048	0.057	0.065	0.074	0.082
0.5	0.006	0.016	0.029	0.041	0.054	0.066	0.077	0.089	0.101	0.112
0.6	0.008	0.022	0.039	0.056	0.072	0.088	0.104	0.120	0.135	0.149
0.7	0.011	0.029	0.053	0.075	0.096	0.117	0.138	0.158	0.177	0.196
0.8	0.014	0.038	0.069	0.098	0.126	0.153	0.179	0.204	0.228	0.251
0.9	0.018	0.049	0.090	0.127	0.162	0.196	0.228	0.259	0.289	0.316
1.0	0.023	0.063	0.116	0.162	0.205	0.246	0.285	0.322	0.357	0.389
1.1	0.029	0.081	0.147	0.203	0.255	0.304	0.349	0.392	0.432	0.468
1.2	0.036	0.102	0.183	0.250	0.311	0.368	0.420	0.467	0.511	0.550
1.3	0.045	0.126	0.224	0.303	0.373	0.437	0.494	0.545	0.591	0.631
1.4	0.055	0.155	0.271	0.362	0.439	0.509	0.570	0.623	0.669	0.709
1.5	0.067	0.188	0.323	0.424	0.508	0.581	0.644	0.697	0.742	0.779
1.6	0.081	0.225	0.378	0.488	0.577	0.652	0.714	0.765	0.806	0.839
1.7	0.097	0.266	0.437	0.554	0.644	0.719	0.778	0.824	0.861	0.889
1.8	0.115	0.310	0.498	0.618	0.708	0.779	0.833	0.873	0.904	0.927
1.9	0.136	0.358	0.559	0.680	0.767	0.832	0.879	0.913	0.937	0.954
2.0	0.159	0.409	0.619	0.738	0.818	0.877	0.916	0.942	0.961	0.973
2.1	0.184	0.461	0.677	0.790	0.863	0.912	0.944	0.964	0.977	0.985
2.2	0.212	0.514	0.730	0.835	0.899	0.940	0.964	0.978	0.987	0.992
2.3	0.242	0.567	0.780	0.874	0.928	0.960	0.978	0.988	0.993	0.996
2.4	0.274	0.619	0.823	0.906	0.950	0.975	0.987	0.993	0.997	0.998
2.5	0.309	0.669	0.861	0.932	0.967	0.985	0.993	0.997	0.998	0.999

Table 2: Power Functions for Charts for Location: Detection Rules One and Two

shift δ	*k=1*	*k=2*	*k=3*	*k=4*	*k=5*	*k=6*	*k=7*	*k=8*	*k=9*	*k=10*
2.6	0.345	0.716	0.893	0.952	0.979	0.991	0.996	0.998	0.999	1.000
2.7	0.382	0.760	0.920	0.967	0.987	0.995	0.998	0.999	1.000	
2.8	0.421	0.799	0.941	0.978	0.992	0.997	0.999	1.000		
2.9	0.460	0.835	0.958	0.986	0.995	0.999	1.000			
3.0	0.500	0.867	0.970	0.991	0.997	0.999				
3.1	0.540	0.894	0.980	0.994	0.999	1.000				
3.2	0.579	0.916	0.986	0.997	0.999					
3.3	0.618	0.935	0.991	0.998	1.000					
3.4	0.655	0.951	0.994	0.999						
3.5	0.691	0.963	0.996	0.999						
3.6	0.726	0.973	0.998	1.000						
3.7	0.758	0.980	0.999							
3.8	0.788	0.986	0.999							
3.9	0.816	0.990	1.000							
4.0	0.841	0.993								
4.1	0.864	0.995								
4.2	0.885	0.997								
4.3	0.903	0.998								
4.4	0.919	0.999								
4.5	0.933	0.999								
4.6	0.945	1.000								
4.7	0.955									
4.8	0.964									
4.9	0.971									
5.0	0.977									
5.1	0.982									
5.2	0.986									
5.3	0.989									
5.4	0.992									
5.5	0.994									
5.6	0.995									
5.7	0.997									
5.8	0.997									
5.9	0.998									
6.0	0.999									

In Figure 4 the curve labeled "Detection Rules One and Two" is found by plotting the last column of values in Table 2 above.

Detection Rules One, Two and Three

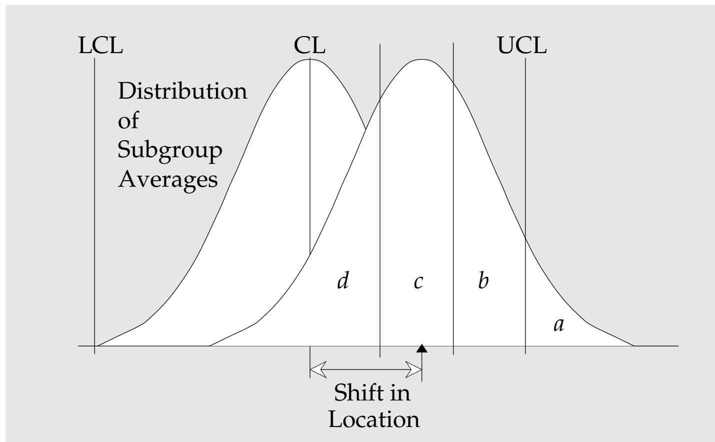

Figure 7: Probabilities for Detection Rules I, II and III

Four values are needed from the probability model when using Detection Rules One, Two and Three together. These are the probability of exceeding the three-sigma limit, a, the probability of being between the two-sigma line and the three-sigma limit, b, the probability of being between the one-sigma line and the two-sigma line, c, and the complement of these three values, $d = 1 - a - b - c$.

Given these basic values, the probability of detecting a shift on the k^{th} subgroup following the shift is p_k where:

$$p_1 = a$$

$$p_2 = b^2 + a(b+c+d)$$

$$p_3 = 2\,ab(c+d) + 2\,b^2(c+d) + a(c+d)^2$$

$$p_4 = 3\,abc^2 + 6\,abcd + 3\,abd^2 + 3\,ac^2d + 3\,acd^2 + ac^3 + ad^3 + 3\,b^2c^2 + 4\,b^2cd + 2\,b^2d^2 + 4\,bc^3 + c^4$$

$$p_5 = 2\,ab^2cd + 2\,b^3cd + 12\,abc^2d + 14\,b^2c^2d + 16\,bc^3d + 4\,ac^3d + 4\,c^4d + ab^2d^2 + b^3d^2 + 12\,abcd^2 + 6\,b^2cd^2 + 6\,ac^2d^2 + 4\,abd^3 + 2\,b^2d^3 + 4\,acd^3 + ad^4$$

$$p_6 = 9\,ab^2cd^2 + 9\,b^3cd^2 + 30\,abc^2d^2 + 20\,b^2c^2d^2 + 10\,ac^3d^2 + 16\,bc^3d^2 + 4\,c^4d^2 + 3\,ab^2d^3 + 3\,b^3d^3 + 20\,abcd^3 + 8\,b^2cd^3 + 10\,ac^2d^3 + 5\,abd^4 + 2\,b^2d^4 + 5\,acd^4 + ad^5$$

$$p_7 = 19\,ab^2c^2d^2 + 18\,b^3c^2d^2 + 24\,abc^3d^2 + 25\,b^2c^3d^2 + 15\,bc^4d^2 + 6\,ac^4d^2 + 3\,c^5d^2 + 24\,ab^2cd^3 + 20\,b^3cd^3 + 60\,abc^2d^3 + 28\,b^2c^2d^3 + 20\,ac^3d^3 + 16\,bc^3d^3 + 4\,c^4d^3 + 6\,ab^2d^4 + 5\,b^3d^4 + 30\,abcd^4 + 10\,b^2cd^4 + 15\,ac^2d^4 + 6\,abd^5 + 2\,b^2d^5 + 6\,acd^5 + ad^6$$

$$p_8 = 3\,ac^5d^2 + 15\,abc^4d^2 + 18\,ab^2c^3d^2 + ab^3c^2d^2 + 2\,c^6d^2 + 12\,bc^5d^2 + 24\,b^2c^4d^2 + 19\,b^3c^3d^2 + b^4c^2d^2 + 22\,ac^4d^3 + 88\,abc^3d^3 + 75\,ab^2c^2d^3 + 4\,ab^3cd^3 + 7\,c^5d^3 + 35\,bc^4d^3 + 69\,b^2c^3d^3 + 66\,b^3c^2d^3 + 4\,b^4cd^3 + 35\,ac^3d^4 + 105\,abc^2d^4 + 50\,ab^2cd^4 + ab^3d^4 + 4\,c^4d^4 + 16\,bc^3d^4 + 38\,b^2c^2d^4 + 35\,b^3cd^4 + b^4d^4 + 21\,ac^2d^5 + 42\,abcd^5 + 10\,ab^2d^5 + 12\,b^2cd^5 + 7\,b^3d^5 + 7acd^6 + 7abd^6 + 2\,b^2d^6 + ad^7$$

$p_9 = ac^6d^2 + 6\,abc^5d^2 + 9\,ab^2c^4d^2 + c^7d^2 + 7\,bc^6d^2 + 15\,b^2c^5d^2 + 9\,b^3c^4d^2 + 18ac^5d^3 +$
$90\,abc^4d^3 + 112ab^2c^3d^3 + 14\,ab^3c^2d^3 + 9\,c^6d^3 + 54\,bc^5d^3 + 114\,b^2c^4d^3 + 102\,b^3c^3d^3 +$
$14\,b^4c^2d^3 + 53\,ac^4d^4 + 212\,abc^3d^4 + 192\,ab^2c^2d^4 + 20\,ab^3cd^4 + 11\,c^5d^4 + 55\,bc^4d^4 +$
$123\,b^2c^3d^4 + 139\,b^3c^2d^4 + 20\,b^4cd^4 + 56\,ac^3d^5 + 168\,abc^2d^5 + 90\,ab^2cd^5 + 4ab^3d^5 +$
$4\,c^4d^5 + 16\,bc^3d^5 + 50\,b^2c^2d^5 + 54\,b^3cd^5 + 4\,b^4d^5 + 28\,ac^2d^6 + 56\,abcd^6 + 15\,ab^2d^6 +$
$14\,b^2cd^6 + 9\,b^3d^6 + 8\,acd^7 + 8\,abd^7 + 2\,b^2d^7 + ad^8$

$p_{10} = 24\,ab^3c^3d^3 + 97\,ab^2c^4d^3 + 60\,abc^5d^3 + 10\,ac^6d^3 + 24\,b^4c^3d^3 + 121\,b^3c^4d^3 +$
$157\,b^2c^5d^3 + 70\,bc^6d^3 + 10\,c^7d^3 + 87\,ab^3c^2d^4 + 393\,ab^2c^3d^4 + 300\,abc^4d^4 +$
$60\,ac^5d^4 + 81\,b^4c^2d^4 + 300\,b^3c^3d^4 + 276\,b^2c^4d^4 + 120\,bc^5d^4 + 20\,c^6d^4 +$
$60\,ab^3cd^5 + 400\,ab^2c^2d^5 + 420\,abc^3d^5 + 105\,ac^4d^5 + 54\,b^4cd^5 + 243\,b^3c^2d^5 +$
$189\,b^2c^3d^5 + 75\,bc^4d^5 + 15\,c^5d^5 + 10\,ab^3d^6 + 147\,ab^2cd^6 + 252\,abc^2d^6 +$
$84\,ac^3d^6 + 9\,b^4d^6 + 77\,b^3cd^6 + 64\,b^2c^2d^6 + 16\,bc^3d^6 + 4\,c^4d^6 + 21\,ab^2d^7 +$
$72\,abcd^7 + 36\,ac^2d^7 + 11\,b^3d^7 + 16\,b^2cd^7 + 9\,abd^8 + 9\,acd^8 + 2\,b^2d^8 + ad^9$

By summing these probabilities we can obtain the probability of detecting a shift within k subgroups following the shift using Detection Rules One, Two and Three. The formulas above completely enumerate the 8108 ways that Detection Rules One, Two, and Three can detect a shift within 10 subgroups following that shift. These formulas are perfectly general—they will work with values of a, b, c, and d obtained from any appropriate probability distribution.

Using the normal distribution, if we express the shift in standard error units as δ, then the probabilities needed for the formulas above will be:

$$\text{Probability of } Z > (3-\delta) = a$$
$$\text{Probability of } (2-\delta) < Z < (3-\delta) = b$$
$$\text{Probability of } (1-\delta) < Z < (2-\delta) = c$$
$$d = 1 - a - b - c$$

where Z is the standard normal variable.

In Figure 4 the curve labeled “Detection Rules One, Two, and Three” is found by plotting the last column of values in Table 3 below.

As in Tables 1 and 2, the shifts shown in Table 3 are expressed in standard error units, *δ*. The values shown are the probabilities of detecting a shift *within k subgroups following the shift.*

Table 3: Power Functions for Location: Detection Rules One, Two, and Three

shift δ	***k=1***	***k=2***	***k=3***	***k=4***	***k=5***	***k=6***	***k=7***	***k=8***	***k=9***	***k=10***
0	0.003	0.006	0.011	0.014	0.016	0.023	0.031	0.038	0.045	0.052
0.1	0.002	0.004	0.008	0.016	0.023	0.029	0.034	0.040	0.045	0.051
0.2	0.003	0.006	0.011	0.017	0.027	0.036	0.045	0.053	0.061	0.070
0.3	0.003	0.009	0.015	0.025	0.040	0.053	0.065	0.077	0.089	0.101
0.4	0.005	0.012	0.021	0.035	0.057	0.076	0.093	0.110	0.127	0.144
0.5	0.006	0.016	0.029	0.048	0.081	0.106	0.130	0.153	0.176	0.198
0.6	0.008	0.022	0.039	0.067	0.111	0.145	0.176	0.206	0.236	0.265
0.7	0.011	0.029	0.053	0.090	0.150	0.193	0.233	0.271	0.308	0.343
0.8	0.014	0.038	0.069	0.120	0.198	0.251	0.299	0.345	0.390	0.430
0.9	0.018	0.049	0.090	0.156	0.254	0.318	0.374	0.426	0.478	0.523
1.0	0.023	0.063	0.116	0.199	0.319	0.392	0.455	0.512	0.568	0.616
1.1	0.029	0.081	0.147	0.250	0.390	0.470	0.538	0.599	0.657	0.704
1.2	0.036	0.102	0.183	0.308	0.466	0.551	0.621	0.681	0.738	0.783
1.3	0.045	0.126	0.224	0.371	0.544	0.629	0.699	0.756	0.810	0.850
1.4	0.055	0.155	0.271	0.438	0.620	0.704	0.769	0.821	0.869	0.902
1.5	0.067	0.188	0.323	0.508	0.692	0.770	0.829	0.874	0.915	0.941
1.6	0.081	0.225	0.378	0.578	0.758	0.828	0.878	0.915	0.948	0.967
1.7	0.097	0.266	0.437	0.646	0.816	0.876	0.917	0.946	0.971	0.984
1.8	0.115	0.310	0.498	0.710	0.864	0.913	0.946	0.967	0.986	0.994
1.9	0.136	0.358	0.559	0.768	0.903	0.942	0.966	0.981	0.994	0.999
2.0	0.159	0.409	0.619	0.819	0.933	0.963	0.980	0.989	0.999	
2.1	0.184	0.461	0.677	0.863	0.956	0.977	0.988	0.995		
2.2	0.212	0.514	0.730	0.899	0.972	0.986	0.994	0.997		
2.3	0.242	0.567	0.780	0.927	0.982	0.992	0.997	0.999		
2.4	0.274	0.619	0.823	0.949	0.990	0.996	0.998	0.999		
2.5	0.309	0.669	0.861	0.966	0.994	0.998	0.999			
2.6	0.345	0.716	0.893	0.977	0.997	0.999				
2.7	0.382	0.760	0.920	0.986	0.998	0.999				
2.8	0.421	0.799	0.941	0.991	0.999					
2.9	0.460	0.835	0.958	0.995						
3.0	0.500	0.867	0.970	0.997						
3.1	0.540	0.894	0.980	0.998						
3.2	0.579	0.916	0.986	0.999						
3.3	0.618	0.935	0.991							
3.4	0.655	0.951	0.994							
3.5	0.691	0.963	0.996							
3.6	0.726	0.973	0.998							
3.7	0.758	0.980	0.999							
3.8	0.788	0.986	0.999							

(see Table 2 for larger shifts when k = 1 or 2)

Detection Rules One, Two, Three and Four

When using Detection Rules One, Two, Three, and Four together five values are needed from the probability model. These are the probability of exceeding the three-sigma limit, a, the probability of being between the two-sigma line and the three-sigma limit, b, the probability of being between the one-sigma line and the two-sigma line, c, the probability of being between the central line and the one-sigma line, d, and the complement of these four values, $e = 1 - a - b - c - d$.

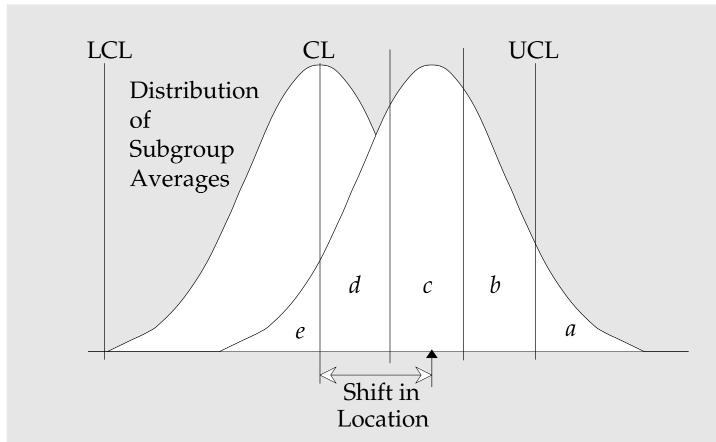

Figure 9.8: Probabilities for Detection Rules I, II, III, and IV

In the following equations a double plus sign ++ will precede terms for Rule Two, a triple plus +++ will precede terms for Rule Three, and a quadruple plus ++++ will precede terms for Rule Four. Given these basic values, the probability of detecting a shift on the k^{th} subgroup following the shift is p_k where:

$p_1 = a \qquad p_2 = a(b+c+d+e) \;{+}{+}\; b^2 \qquad p_3 = 2\,ab(c+d+e) + a(c+d+e)^2 \;{+}{+}\; 2\,b^2(c+d+e)$

$$p_4 = 3\,abc^2 + 6\,abc(d+e) + 3\,ab(d+e)^2 + 3\,ac^2(d+e) + 3\,ac(d+e)^2 + ac^3 + a(d+e)^3$$
$${+}{+}\; 2\,b^2c^2 + 4\,b^2c(d+e) + 2\,b^2(d+e)^2 \;{+}{+}{+}\; b^2c^2 + 4\,bc^3 + c^4$$

$$p_5 = 2\,ab^2c(d+e) + 12\,abc^2(d+e) + 4\,ac^3(d+e) + 12\,abc(d+e)^2 + ab^2(d+e)^2 + 6\,ac^2(d+e)^2 +$$
$$4\,ab(d+e)^3 + 4\,ac(d+e)^3 + a(d+e)^4 \;{+}{+}\; 2\,b^3c(d+e) + 6\,b^2c^2(d+e) + b^3(d+e)^2 + 6\,b^2c(d+e)^2 +$$
$$2\,b^2(d+e)^3 \;{+}{+}{+}\; 8\,b^2c^2(d+e) + 16\,bc^3(d+e) + 4\,c^4(d+e)$$

$$p_6 = 9\,ab^2c(d+e)^2 + 30\,abc^2(d+e)^2 + 10\,ac^3(d+e)^2 + 3\,ab^2(d+e)^3 + 20\,abc(d+e)^3 +$$
$$10\,ac^2(d+e)^3 + 5\,ab(d+e)^4 + 5\,ac(d+e)^4 + a(d+e)^5 \;{+}{+}\; 3\,b^3(d+e)^3 + 9\,b^3c(d+e)^2 +$$
$$12\,b^2c^2(d+e)^2 + 8\,b^2c(d+e)^3 + 2\,b^2(d+e)^4 \;{+}{+}{+}\; 16\,bc^3(d+e)^2 + 8\,b^2c^2(d+e)^2 + 4\,c^4(d+e)^2$$

$$p_7 = 19\,ab^2c^2(d+e)^2 + 24\,abc^3(d+e)^2 + 6\,ac^4(d+e)^2 + 24\,ab^2c(d+e)^3 + 60\,abc^2(d+e)^3 + 20\,ac^3(d+e)^3 +$$
$$6\,ab^2(d+e)^4 + 30\,abc(d+e)^4 + 15\,ac^2(d+e)^4 + 6\,ab(d+e)^5 + 6\,ac(d+e)^5 + a(d+e)^6$$
$${+}{+}\; 5\,b^3(d+e)^4 + 20\,b^3c(d+e)^3 + 10\,b^2c(d+e)^4 + 2\,b^2(d+e)^5 + 9\,b^2c^3(d+e)^2 + 20\,b^2c^2(d+e)^3 +$$
$$16\,b^3c^2(d+e)^2 \;{+}{+}{+}\; 3\,c^5(d+e)^2 + 4\,c^4(d+e)^3 + 15\,bc^4(d+e)^2 + 16\,bc^3(d+e)^3 + 2\,b^3c^2(d+e)^2 +$$
$$16\,b^2c^3(d+e)^2 + 8\,b^2c^2(d+e)^3$$

p_8 = $ae^7 + 7\,ade^6 + 21\,ad^2e^5 + 35\,ad^3e^4 + 35\,ad^4e^3 + 21\,ad^5e^2 + 7\,ad^6e + ad^7 + 7\,ace^6 +$
$42\,acde^5 + 105\,acd^2e^4 + 140\,acd^3e^3 + 105\,acd^4e^2 + 42\,acd^5e + 7\,acd^6 + 21\,ac^2e^5 +$
$105\,ac^2de^4 + 210\,ac^2d^2e^3 + 210\,ac^2d^3e^2 + 105\,ac^2d^4e + 21\,ac^2d^5 + 35\,ac^3e^4 +$
$140\,ac^3de^3 + 210\,ac^3d^2e^2 + 140\,ac^3d^3e + 35\,ac^3d^4 + 22\,ac^4e^3 + 66\,ac^4de^2 +$
$66\,ac^4d^2e + 22\,ac^4d^3 + 3\,ac^5e^2 + 6\,ac^5de + 3\,ac^5d^2 + 7\,abe^6 + 42\,abde^5 + 105\,abd^2e^4 +$
$140\,abd^3e^3 + 105\,abd^4e^2 + 42\,abd^5e + 7\,abd^6 + 42\,abce^5 + 210\,abcde^4 + 420\,abcd^2e^3 +$
$420\,abcd^3e^2 + 210\,abcd^4e + 42\,abcd^5 + 105\,abc^2e^4 + 420\,abc^2de^3 + 630\,abc^2d^2e^2 +$
$420\,abc^2d^3e + 105\,abc^2d^4 + 88\,abc^3e^3 + 264\,abc^3de^2 + 264\,abc^3d^2e + 88\,abc^3d^3 +$
$15\,abc^4e^2 + 30\,abc^4de + 15\,abc^4d^2 + 10\,ab^2e^5 + 50\,ab^2de^4 + 100\,ab^2d^2e^3 +$
$100\,ab^2d^3e^2 + 50\,ab^2d^4e + 10\,ab^2d^5 + 50\,ab^2ce^4 + 200\,ab^2cde^3 + 300\,ab^2cd^2e^2 +$
$200\,ab^2cd^3e + 50\,ab^2cd^4 + 75\,ab^2c^2e^3 + 225\,ab^2c^2de^2 + 225\,ab^2c^2d^2e + 75\,ab^2c^2d^3 +$
$18\,ab^2c^3e^2 + 36\,ab^2c^3de + 18\,ab^2c^3d^2 + ab^3e^4 + 4\,ab^3de^3 + 6\,ab^3d^2e^2 + 4\,ab^3d^3e + ab^3d^4 +$
$4\,ab^3ce^3 + 12\,ab^3cde^2 + 12\,ab^3cd^2e + 4\,ab^3cd^3 + ab^3c^2e^2 + 2\,ab^3c^2de + ab^3c^2d^2$
$++ 2\,b^2e^6 + 12\,b^2de^5 + 30\,b^2d^2e^4 + 40\,b^2d^3e^3 + 30\,b^2d^4e^2 + 12\,b^2d^5e + 2\,b^2d^6 +$
$12\,b^2ce^5 + 60\,b^2cde^4 + 120\,b^2cd^2e^3 + 120\,b^2cd^3e^2 + 60\,b^2cd^4e + 12\,b^2cd^5 + 30\,b^2c^2e^4 +$
$120\,b^2c^2de^3 + 180\,b^2c^2d^2e^2 + 120\,b^2c^2d^3e + 30\,b^2c^2d^4 + 29\,b^2c^3e^3 + 87\,b^2c^3de^2 +$
$87\,b^2c^3d^2e + 29\,b^2c^3d^3 + 6\,b^2c^4e^2 + 12\,b^2c^4de + 6\,b^2c^4d^2 + 7\,b^3e^5 + 35\,b^3de^4 +$
$70\,b^3d^2e^3 + 70\,b^3d^3e^2 + 35\,b^3d^4e + 7\,b^3d^5 + 35\,b^3ce^4 + 140\,b^3cde^3 + 210\,b^3cd^2e^2 +$
$140\,b^3cd^3e + 35\,b^3cd^4 + 56\,b^3c^2e^3 + 168\,b^3c^2de^2 + 168\,b^3c^2d^2e + 56\,b^3c^2d^3 +$
$15\,b^3c^3e^2 + 30\,b^3c^3de + 15\,b^3c^3d^2 + b^4e^4 + 4\,b^4de^3 + 6\,b^4d^2e^2 + 4\,b^4d^3e + b^4d^4 +$
$4\,b^4ce^3 + 12\,b^4cde^2 + 12\,b^4cd^2e + 4\,b^4cd^3 + b^4c^2e^2 + 2\,b^4c^2de + b^4c^2d^2$
$+++ 4\,c^4e^4 + 16\,c^4de^3 + 24\,c^4d^2e^2 + 16\,c^4d^3e + 4\,c^4d^4 + 7\,c^5e^3 + 21\,c^5de^2 + 21\,c^5d^2e +$
$7\,c^5d^3 + 2\,c^6e^2 + 4\,c^6de + 2\,c^6d^2 + 16\,bc^3e^4 + 64\,bc^3de^3 + 96\,bc^3d^2e^2 + 64\,bc^3d^3e +$
$16\,bc^3d^4 + 35\,bc^4e^3 + 105\,bc^4de^2 + 105\,bc^4d^2e + 35\,bc^4d^3 + 12\,bc^5e^2 + 24\,bc^5de +$
$12\,bc^5d^2 + 6\,b^2c^2e^4 + 24\,b^2c^2de^3 + 36\,b^2c^2d^2e^2 + 24\,b^2c^2d^3e + 6\,b^2c^2d^4 + 36\,b^2c^3e^3 +$
$108\,b^2c^3de^2 + 108\,b^2c^3d^2e + 36\,b^2c^3d^3 + 17\,b^2c^4e^2 + 34\,b^2c^4de + 17\,b^2c^4d^2 +$
$8\,b^3c^2e^3 + 24\,b^3c^2de^2 + 24\,b^3c^2d^2e + 8\,b^3c^2d^3 + 3\,b^3c^3e^2 + 6\,b^3c^3de + 3\,b^3c^3d^2$
$++++ d^8 + 8\,cd^7 + 28\,c^2d^6 + 56\,c^3d^5 + 53\,c^4d^4 + 18\,c^5d^3 + c^6d^2 + 8\,bd^7 + 56\,bcd^6 +$
$168\,bc^2d^5 + 212\,bc^3d^4 + 90\,bc^4d^3 + 6\,bc^5d^2 + 15\,b^2d^6 + 90\,b^2cd^5 + 192\,b^2c^2d^4 +$
$112\,b^2c^3d^3 + 9\,b^2c^4d^2 + 4\,b^3d^5 + 20\,b^3cd^4 + 14\,b^3c^2d^3$

$$
\begin{aligned}
p_9 \;=\; & ae^8 + 8\,ade^7 + 28\,ad^2e^6 + 56\,ad^3e^5 + 70\,ad^4e^4 + 56\,ad^5e^3 + 28\,ad^6e^2 + 8\,ad^7e + 8\,ace^7 + \\
& 56\,acde^6 + 168\,acd^2e^5 + 280\,acd^3e^4 + 280\,acd^4e^3 + 168\,acd^5e^2 + 56\,acd^6e + 28\,ac^2e^6 + \\
& 168\,ac^2de^5 + 420\,ac^2d^2e^4 + 560\,ac^2d^3e^3 + 420\,ac^2d^4e^2 + 168\,ac^2d^5e + 56\,ac^3e^5 + 280\,ac^3de^4 + \\
& 560\,ac^3d^2e^3 + 560\,ac^3d^3e^2 + 280\,ac^3d^4e + 53\,ac^4e^4 + 212\,ac^4de^3 + 318\,ac^4d^2e^2 + 212\,ac^4d^3e + \\
& 18\,ac^5e^3 + 54\,ac^5de^2 + 54\,ac^5d^2e + ac^6e^2 + 2\,ac^6de + 8\,abe^7 + 56\,abde^6 + 168\,abd^2e^5 + \\
& 280\,abd^3e^4 + 280\,abd^4e^3 + 168\,abd^5e^2 + 56\,abd^6e + 56\,abce^6 + 336\,abcde^5 + 840\,abcd^2e^4 + \\
& 1120\,abcd^3e^3 + 840\,abcd^4e^2 + 336\,abcd^5e + 168\,abc^2e^5 + 840\,abc^2de^4 + 1680\,abc^2d^2e^3 + \\
& 1680\,abc^2d^3e^2 + 840\,abc^2d^4e + 212\,abc^3e^4 + 848\,abc^3de^3 + 1272\,abc^3d^2e^2 + 848\,abc^3d^3e + \\
& 90\,abc^4e^3 + 270\,abc^4de^2 + 270\,abc^4d^2e + 6\,abc^5e^2 + 12\,abc^5de + 15\,ab^2e^6 + 90\,ab^2de^5 + \\
& 225\,ab^2d^2e^4 + 300\,ab^2d^3e^3 + 225\,ab^2d^4e^2 + 90\,ab^2d^5e + 90\,ab^2ce^5 + 450\,ab^2cde^4 + \\
& 900\,ab^2cd^2e^3 + 900\,ab^2cd^3e^2 + 450\,ab^2cd^4e + 192\,ab^2c^2e^4 + 768\,ab^2c^2de^3 + 1152\,ab^2c^2d^2e^2 + \\
& 768\,ab^2c^2d^3e + 112\,ab^2c^3e^3 + 336\,ab^2c^3de^2 + 336\,ab^2c^3d^2e + 9\,ab^2c^4e^2 + 18\,ab^2c^4de + 4\,ab^3e^5 + \\
& 20\,ab^3de^4 + 40\,ab^3d^2e^3 + 40\,ab^3d^3e^2 + 20\,ab^3d^4e + 20\,ab^3ce^4 + 80\,ab^3cde^3 + 120\,ab^3cd^2e^2 + \\
& 80\,ab^3cd^3e + 14\,ab^3c^2e^3 + 42\,ab^3c^2de^2 + 42\,ab^3c^2d^2e \\
& ++\,2\,b^2e^7 + 14\,b^2de^6 + 42\,b^2d^2e^5 + 70\,b^2d^3e^4 + 70\,b^2d^4e^3 + 42\,b^2d^5e^2 + 14\,b^2d^6e + \\
& 14\,b^2ce^6 + 84\,b^2cde^5 + 210\,b^2cd^2e^4 + 280\,b^2cd^3e^3 + 210\,b^2cd^4e^2 + 84\,b^2cd^5e + 42\,b^2c^2e^5 + \\
& 210\,b^2c^2de^4 + 420\,b^2c^2d^2e^3 + 420\,b^2c^2d^3e^2 + 210\,b^2c^2d^4e + 59\,b^2c^3e^4 + 236\,b^2c^3de^3 + \\
& 354\,b^2c^3d^2e^2 + 236\,b^2c^3d^3e + 28\,b^2c^4e^3 + 84\,b^2c^4de^2 + 84\,b^2c^4d^2e + 2\,b^2c^5e^2 + 4\,b^2c^5de + \\
& 9\,b^3e^6 + 54\,b^3de^5 + 135\,b^3d^2e^4 + 180\,b^3d^3e^3 + 135\,b^3d^4e^2 + 54\,b^3d^5e + 54\,b^3ce^5 + 270\,b^3cde^4 + \\
& 540\,b^3cd^2e^3 + 540\,b^3cd^3e^2 + 270\,b^3cd^4e + 121\,b^3c^2e^4 + 484\,b^3c^2de^3 + 726\,b^3c^2d^2e^2 + \\
& 484\,b^3c^2d^3e + 74\,b^3c^3e^3 + 222\,b^3c^3de^2 + 222\,b^3c^3d^2e + 6\,b^3c^4e^2 + 12\,b^3c^4de + 4\,b^4e^5 + \\
& 20\,b^4de^4 + 40\,b^4d^2e^3 + 40\,b^4d^3e^2 + 20\,b^4d^4e + 20\,b^4ce^4 + 80\,b^4cde^3 + 120\,b^4cd^2e^2 + 80\,b^4cd^3e + \\
& 14\,b^4c^2e^3 + 42\,b^4c^2de^2 + 42\,b^4c^2d^2e \\
& +++\,4\,c^4e^5 + 20\,c^4de^4 + 40\,c^4d^2e^3 + 40\,c^4d^3e^2 + 20\,c^4d^4e + 11\,c^5e^4 + 44\,c^5de^3 + 66\,c^5d^2e^2 + \\
& 44\,c^5d^3e + 9\,c^6e^3 + 27\,c^6de^2 + 27\,c^6d^2e + c^7e^2 + 2\,c^7de + 16\,bc^3e^5 + 80\,bc^3de^4 + 160\,bc^3d^2e^3 + \\
& 160\,bc^3d^3e^2 + 80\,bc^3d^4e + 55\,bc^4e^4 + 220\,bc^4de^3 + 330\,bc^4d^2e^2 + 220\,bc^4d^3e + 54\,bc^5e^3 + \\
& 162\,bc^5de^2 + 162\,bc^5d^2e + 7\,bc^6e^2 + 14\,bc^6de + 8\,b^2c^2e^5 + 40\,b^2c^2de^4 + 80\,b^2c^2d^2e^3 + \\
& 80\,b^2c^2d^3e^2 + 40\,b^2c^2d^4e + 64\,b^2c^3e^4 + 256\,b^2c^3de^3 + 384\,b^2c^3d^2e^2 + 256\,b^2c^3d^3e + 86\,b^2c^4e^3 + \\
& 258\,b^2c^4de^2 + 258\,b^2c^4d^2e + 13\,b^2c^5e^2 + 26\,b^2c^5de + 18\,b^3c^2e^4 + 72\,b^3c^2de^3 + 108\,b^3c^2d^2e^2 + \\
& 72\,b^3c^2d^3e + 28\,b^3c^3e^3 + 84\,b^3c^3de^2 + 84\,b^3c^3d^2e + 3\,b^3c^4e^2 + 6\,b^3c^4de \\
& ++++\,d^8e + 8\,cd^7e + 28\,c^2d^6e + 56\,c^3d^5e + 53\,c^4d^4e + 18\,c^5d^3e + c^6d^2e + 8\,bd^7e + 56\,bcd^6e + \\
& 168\,bc^2d^5e + 212\,bc^3d^4e + 90\,bc^4d^3e + 6\,bc^5d^2e + 15\,b^2d^6e + 90\,b^2cd^5e + 192\,b^2c^2d^4e + \\
& 112\,b^2c^3d^3e + 9\,b^2c^4d^2e + 4\,b^3d^5e + 20\,b^3cd^4e + 14\,b^3c^2d^3e
\end{aligned}
$$

$$p_{10} = ae^9 + 9\,ade^8 + 36\,ad^2e^7 + 84\,ad^3e^6 + 126\,ad^4e^5 + 126\,ad^5e^4 + 84\,ad^6e^3 + 36\,ad^7e^2 + 7\,ad^8e +$$
$$9\,ace^8 + 72\,acde^7 + 252\,acd^2e^6 + 504\,acd^3e^5 + 630\,acd^4e^4 + 504\,acd^5e^3 + 252\,acd^6e^2 + 56\,acd^7e +$$
$$36\,ac^2e^7 + 252\,ac^2de^6 + 756\,ac^2d^2e^5 + 1260\,ac^2d^3e^4 + 1260\,ac^2d^4e^3 + 756\,ac^2d^5e^2 + 196\,ac^2d^6e +$$
$$84\,ac^3e^6 + 504\,ac^3de^5 + 1260\,ac^3d^2e^4 + 1680\,ac^3d^3e^3 + 1260\,ac^3d^4e^2 + 392\,ac^3d^5e + 105\,ac^4e^5 +$$
$$525\,ac^4de^4 + 1050\,ac^4d^2e^3 + 1050\,ac^4d^3e^2 + 419\,ac^4d^4e + 60\,ac^5e^4 + 240\,ac^5de^3 + 360\,ac^5d^2e^2 +$$
$$204\,ac^5d^3e + 10\,ac^6e^3 + 30\,ac^6de^2 + 28\,ac^6d^2e + 9\,abe^8 + 72\,abde^7 + 252\,abd^2e^6 + 504\,abd^3e^5 +$$
$$630\,abd^4e^4 + 504\,abd^5e^3 + 252\,abd^6e^2 + 56\,abd^7e + 72\,abce^7 + 504\,abcde^6 + 1512\,abcd^2e^5 +$$
$$2520\,abcd^3e^4 + 2520\,abcd^4e^3 + 1512\,abcd^5e^2 + 392\,abcd^6e + 252\,abc^2e^6 + 1512\,abc^2de^5 +$$
$$3780\,abc^2d^2e^4 + 5040\,abc^2d^3e^3 + 3780\,abc^2d^4e^2 + 1176\,abc^2d^5e + 420\,abc^3e^5 + 2100\,abc^3de^4 +$$
$$4200\,abc^3d^2e^3 + 4200\,abc^3d^3e^2 + 1676\,abc^3d^4e + 300\,abc^4e^4 + 1200\,abc^4de^3 + 1800\,abc^4d^2e^2 +$$
$$1020\,abc^4d^3e + 60\,abc^5e^3 + 180\,abc^5de^2 + 168\,abc^5d^2e + 21\,ab^2e^7 + 147\,ab^2de^6 + 441\,ab^2d^2e^5 +$$
$$735\,ab^2d^3e^4 + 735\,ab^2d^4e^3 + 441\,ab^2d^5e^2 + 117\,ab^2d^6e + 147\,ab^2ce^6 + 882\,ab^2cde^5 +$$
$$2205\,ab^2cd^2e^4 + 2940\,ab^2cd^3e^3 + 2205\,ab^2cd^4e^2 + 702\,ab^2cd^5e + 400\,ab^2c^2e^5 + 2000\,ab^2c^2de^4 +$$
$$4000\,ab^2c^2d^2e^3 + 4000\,ab^2c^2d^3e^2 + 1616\,ab^2c^2d^4e + 393\,ab^2c^3e^4 + 1572\,ab^2c^3de^3 +$$
$$2358\,ab^2c^3d^2e^2 + 1348\,ab^2c^3d^3e + 97\,ab^2c^4e^3 + 291\,ab^2c^4de^2 + 273\,ab^2c^4d^2e + 10\,ab^3e^6 +$$
$$60\,ab^3de^5 + 150\,ab^3d^2e^4 + 200\,ab^3d^3e^3 + 150\,ab^3d^4e^2 + 52\,ab^3d^5e + 60\,ab^3ce^5 + 300\,ab^3cde^4 +$$
$$600\,ab^3cd^2e^3 + 600\,ab^3cd^3e^2 + 260\,ab^3cd^4e + 187\,ab^3c^2e^4 + 348\,ab^3c^2de^3 + 522\,ab^3c^2d^2e^2 +$$
$$320\,ab^3c^2d^3e + 24\,ab^3c^3e^3 + 72\,ab^3c^3de^2 + 72\,ab^3c^3d^2e$$
$$++\ 2\,b^2e^8 + 16\,b^2de^7 + 56\,b^2d^2e^6 + 112\,b^2d^3e^5 + 140\,b^2d^4e^4 + 112\,b^2d^5e^3 + 56\,b^2d^6e^2 + 13\,b^2d^7e +$$
$$16\,b^2ce^7 + 112\,b^2cde^6 + 336\,b^2cd^2e^5 + 560\,b^2cd^3e^4 + 560\,b^2cd^4e^3 + 336\,b^2cd^5e^2 + 91\,b^2cd^6e +$$
$$56\,b^2c^2e^6 + 336\,b^2c^2de^5 + 840\,b^2c^2d^2e^4 + 1120\,b^2c^2d^3e^3 + 840\,b^2c^2d^4e^2 + 273\,b^2c^2d^5e +$$
$$101\,b^2c^3e^5 + 505\,b^2c^3de^4 + 1010\,b^2c^3d^2e^3 + 1010\,b^2c^3d^3e^2 + 415\,b^2c^3d^4e + 79\,b^2c^4e^4 +$$
$$316\,b^2c^4de^3 + 474\,b^2c^4d^2e^2 + 273\,b^2c^4d^3e + 17\,b^2c^5e^3 + 51\,b^2c^5de^2 + 48\,b^2c^5d^2e + 11\,b^3e^7 +$$
$$77\,b^3de^6 + 231\,b^3d^2e^5 + 385\,b^3d^3e^4 + 385\,b^3d^4e^3 + 231\,b^3d^5e^2 + 63\,b^3d^6e + 77\,b^3ce^6 +$$
$$462\,b^3cde^5 + 1155\,b^3cd^2e^4 + 1540\,b^3cd^3e^3 + 1155\,b^3cd^4e^2 + 378\,b^3cd^5e + 217\,b^3c^2e^5 +$$
$$1085\,b^3c^2de^4 + 2170\,b^3c^2d^2e^3 + 2170\,b^3c^2d^3e^2 + 895\,b^3c^2d^4e + 222\,b^3c^3e^4 + 888\,b^3c^3de^3 +$$
$$1332\,b^3c^3d^2e^2 + 772\,b^3c^3d^3e + 56\,b^3c^4e^3 + 168\,b^3c^4de^2\ \ 159\,b^3c^4d^2e + 9\,b^4e^6 + 54\,b^4de^5 +$$
$$135\,b^4d^2e^4 + 180\,b^4d^3e^3 + 135\,b^4d^4e^2 + 47\,b^4d^5e + 54\,b^4ce^5 + 270\,b^4cde^4 + 540\,b^4cd^2e^3 +$$
$$540\,b^4cd^3e^2 + 235\,b^4cd^4e + 79\,b^4c^2e^4 + 316\,b^4c^2de^3 + 474\,b^4c^2d^2e^2 + 291\,b^4c^2d^3e + 21\,b^4c^3e^3 +$$
$$63\,b^4c^3de^2 + 63\,b^4c^3d^2e$$
$$+++\ 4\,c^4e^6 + 24\,c^4de^5 + 60\,c^4d^2e^4 + 80\,c^4d^3e^3 + 60\,c^4d^4e^2 + 19\,c^4d^5e + 15\,c^5e^5 + 75\,c^5de^4 +$$

$150\ c^5d^2e^3 + 150\ c^5d^3e^2 + 61\ c^5d^4e + 20\ c^6e^4 + 80\ c^6de^3 + 120\ c^6d^2e^2 + 68\ c^6d^3e + 10\ c^7e^3 +$
$30\ c^7de^2 + 28\ c^7d^2e + 16\ bc^3e^6 + 96\ bc^3de^5 + 240\ bc^3d^2e^4 + 320\ bc^3d^3e^3 + 240\ bc^3d^4e^2 +$
$76\ bc^3d^5e + 75\ bc^4e^5 + 375\ bc^4de^4 + 750\ bc^4d^2e^3 + 750\ bc^4d^3e^2 + 305\ bc^4d^4e + 120\ bc^5e^4 +$
$480\ bc^5de^3 + 720\ bc^5d^2e^2 + 408\ bc^5d^3e + 70\ bc^6e^3 + 210\ bc^6de^2 + 196\ bc^6d^2e + 14\ b^2c^2e^6 +$
$84\ b^2c^2de^5 + 210\ b^2c^2d^2e^4 + 280\ b^2c^2d^3e^3 + 210\ b^2c^2d^4e^2 + 67\ b^2c^2d^5e + 111\ b^2c^3e^5 +$
$555\ b^2c^3de^4 + 1110\ b^2c^3d^2e^3 + 1110\ b^2c^3d^3e^2 + 453\ b^2c^3d^4e + 229\ b^2c^4e^4 + 916\ b^2c^4de^3 +$
$1374\ b^2c^4d^2e^2 + 782\ b^2c^4d^3e + 157\ b^2c^5e^3 + 471\ b^2c^5de^2 + 441\ b^2c^5d^2e + 2\ b^3ce^6 +$
$12\ b^3cde^5 + 30\ b^3cd^2e^4 + 40\ b^3cd^3e^3 + 30\ b^3cd^4e^2 + 10\ b^3cd^5e + 57\ b^3c^2e^5 + 285\ b^3c^2de^4 +$
$570\ b^3c^2d^2e^3 + 570\ b^3c^2d^3e^2 + 235\ b^3c^2d^4e + 153\ b^3c^3e^4 + 612\ b^3c^3de^3 + 918\ b^3c^3d^2e^2 +$
$531\ b^3c^3d^3e + 121\ b^3c^4e^3 + 363\ b^3c^4de^2 + 345\ b^3c^4d^2e + 6\ b^4ce^5 + 30\ b^4cde^4 + 60\ b^4cd^2e^3 +$
$60\ b^4cd^3e^2 + 26\ b^4cd^4e + 24\ b^4c^2e^4 + 96\ b^4c^2de^3 + 144\ b^4c^2d^2e^2 + 89\ b^4c^2d^3e + 24\ b^4c^3e^3 +$
$72\ b^4c^3de^2 + 72\ b^4c^3d^2e$

$++++\ d^8e^2 + d^9e + 8\ cd^7e^2 + 9\ cd^8e + 28\ c^2d^6e^2 + 36\ c^2d^7e + 56\ c^3d^5e^2 + 84\ c^3d^6e + 53\ c^4d^4e^2 +$
$108\ c^4d^5e + 18\ c^5d^3e^2 + 68\ c^5d^4e + c^6d^2e^2 + 16\ c^6d^3e + 8\ bd^7e^2 + 9\ bd^8e + 56\ bcd^6e^2 +$
$72\ bcd^7e + 168\ bc^2d^5e^2 + 252\ bc^2d^6e + 212\ bc^3d^4e^2 + 432\ bc^3d^5e + 90\ bc^4d^3e^2 + 340\ bc^4d^4e +$
$6\ bc^5d^2e^2 + 96\ bc^5d^3e + 15\ b^2d^6e^2 + 22\ b^2d^7e + 90\ b^2cd^5e^2 + 154\ b^2cd^6e + 192\ b^2c^2d^4e^2 +$
$427\ b^2c^2d^5e + 112\ b^2c^3d^3e^2 + 467\ b^2c^3d^4e + 9\ b^2c^4d^2e^2 + 166\ b^2c^4d^3e + 4\ b^3d^5e^2 + 14\ b^3d^6e +$
$20\ b^3cd^4e^2 + 84\ b^3cd^5e + 14\ b^3c^2d^3e^2 + 151\ b^3c^2d^4e + 72\ b^3c^3d^3e + 6\ b^4cd^4e + 3\ b^4c^2d^3e$

By summing these probabilities we can obtain the probability of detecting a shift within k subgroups following the shift using Detection Rules One, Two, Three, and Four. The formulas above completely enumerate the 204,267 ways that Detection Rules One, Two, Three, and Four can detect a shift within 10 subgroups following that shift. These formulas are perfectly general—they will work with values of a, b, c, d and e obtained from any appropriate probability distribution. Using the normal distribution, if we express the shift in standard error units as δ, then the probabilities needed for the formulas above will be:

$$\text{Probability of } Z > (3-\delta) = a$$
$$\text{Probability of } (2-\delta) < Z < (3-\delta) = b$$
$$\text{Probability of } (1-\delta) < Z < (2-\delta) = c$$
$$\text{Probability of } (0-\delta) < Z < (1-\delta) = d$$
$$e = 1 - a - b - c - d$$

where Z is the standard normal variable.

As in the preceding tables, the shifts shown in Table 4 are expressed in standard error units, δ. The values shown are the probabilities of detecting a shift *within k subgroups following the shift*.

Table 4: Power Functions for Location: Detection Rules One, Two, Three, and Four

shift δ	***k=1***	***k=2***	***k=3***	***k=4***	***k=5***	***k=6***	***k=7***	***k=8***	***k=9***	***k=10***
0	0.003	0.006	0.011	0.016	0.025	0.032	0.040	0.054	0.065	0.075
0.1	0.002	0.004	0.008	0.012	0.018	0.024	0.030	0.042	0.050	0.058
0.2	0.003	0.006	0.011	0.017	0.027	0.036	0.045	0.063	0.076	0.088
0.3	0.003	0.009	0.015	0.025	0.040	0.053	0.065	0.093	0.111	0.129
0.4	0.005	0.012	0.021	0.035	0.057	0.076	0.093	0.134	0.159	0.183
0.5	0.006	0.016	0.029	0.048	0.081	0.106	0.130	0.187	0.219	0.250
0.6	0.008	0.022	0.039	0.067	0.111	0.145	0.176	0.252	0.291	0.329
0.7	0.011	0.029	0.053	0.090	0.150	0.193	0.233	0.328	0.374	0.419
0.8	0.014	0.038	0.069	0.120	0.198	0.251	0.299	0.413	0.464	0.514
0.9	0.018	0.049	0.090	0.156	0.254	0.318	0.374	0.503	0.557	0.609
1.0	0.023	0.063	0.116	0.199	0.319	0.392	0.455	0.594	0.648	0.699
1.1	0.029	0.081	0.147	0.250	0.390	0.470	0.538	0.681	0.732	0.779
1.2	0.036	0.102	0.183	0.308	0.466	0.551	0.621	0.759	0.804	0.846
1.3	0.045	0.126	0.224	0.371	0.544	0.629	0.699	0.825	0.864	0.898
1.4	0.055	0.155	0.271	0.438	0.620	0.704	0.769	0.879	0.910	0.936
1.5	0.067	0.188	0.323	0.508	0.692	0.770	0.829	0.920	0.943	0.962
1.6	0.081	0.225	0.378	0.578	0.758	0.828	0.878	0.950	0.966	0.979
1.7	0.097	0.266	0.437	0.646	0.816	0.876	0.917	0.970	0.981	0.989
1.8	0.115	0.310	0.498	0.710	0.864	0.913	0.946	0.983	0.990	0.995
1.9	0.136	0.358	0.559	0.768	0.903	0.942	0.966	0.991	0.995	0.998
2.0	0.159	0.409	0.619	0.819	0.933	0.963	0.980	0.995	0.998	0.999
2.1	0.184	0.461	0.677	0.863	0.956	0.977	0.988	0.998	0.999	1.000
2.2	0.212	0.514	0.730	0.899	0.972	0.986	0.994	0.999	1.000	
2.3	0.242	0.567	0.780	0.927	0.982	0.992	0.997	1.000		
2.4	0.274	0.619	0.823	0.949	0.990	0.996	0.998			
2.5	0.309	0.669	0.861	0.966	0.994	0.998	0.999			
2.6	0.345	0.716	0.893	0.977	0.997	0.999	1.000			
2.7	0.382	0.760	0.920	0.986	0.998	0.999				
2.8	0.421	0.799	0.941	0.991	0.999	1.000				
2.9	0.460	0.835	0.958	0.995	1.000					
3.0	0.500	0.867	0.970	0.997						
3.1	0.540	0.894	0.980	0.998						
3.2	0.579	0.916	0.986	0.999						
3.3	0.618	0.935	0.991	1.000						
3.4	0.655	0.951	0.994							
3.5	0.691	0.963	0.996							
3.6	0.726	0.973	0.998							
3.7	0.758	0.980	0.999							
3.8	0.788	0.986	0.999							

(see Table 2 for larger shifts when *k* = 1 or 2)

In Figure 4 the curve labeled "Detection Rules One, Two, Three, and Four" is found by plotting the last column of values in Table 4 above.

Detection Rules One and Four

For routine practice Detection Rules One and Four are often used together. In this case we find that Rule Four adds more than it does when all four rules are used together. The probabilities of detecting a shift on the first seven subgroups following a shift are the same as for Rule One alone. The sum of these probabilities is neatly expressed by the well known formula:

$$\textit{Probability of detecting shift within k subgroups following shift} = \sum_{i=1}^{k} p_i$$
$$= \sum_{i=1}^{k} a\ (1 - a)^{i-1}$$
$$= 1 - (1 - a)^k$$

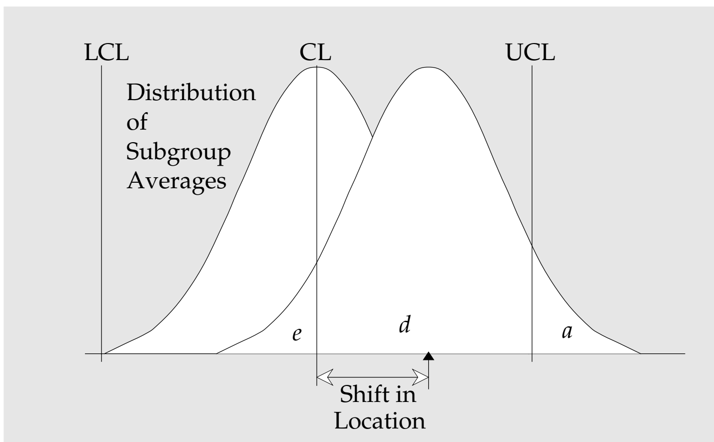

Figure 9: Probabilities for Detection Rules One and Four

Figure 9 shows the labels for the different probabilities associated with the use of Detection Rules One and Four. On the eighth, ninth, and tenth subgroups there is a possibility of detecting a small shift with Rule Four. The appropriate equations for p_8, p_9, and p_{10} are:

$$p_8 = a(1\text{-}a)^7 + d^8$$

$$p_9 = 7\,ad^7e + 28\,ad^6e^2 + 21\,ad^5e^3 + 15\,ad^4e^4 + 21\,ad^3e^5 + 28\,ad^2e^6 + 7\,ade^7 + d^8e$$

$$p_{10} = 7\,ad^8e + 36\,ad^7e^2 + 28\,ad^6e^3 + 21\,ad^5e^4 + 21\,ad^4e^5 + 28\,ad^3e^6 + 36\,ad^2e^7 + 7\,ade^8 + d^8e^2$$

By summing these probabilities we can obtain the probability of detecting a shift within *k* subgroups following the shift using Detection Rules One and Four. The formulas above completely enumerate the 343 ways that Detection Rules One and Four can detect a shift within 10 subgroups following that shift. These formulas are perfectly general—they will work with values of *a, d,* and *e* obtained from any appropriate probability distribution. Using the normal distribution, if we express the shift in standard error units as δ, then the probabilities needed for the formulas above will be:

$$\textit{Probability of } Z > (3 - \delta) = a$$
$$\textit{Probability of } (0 - \delta) < Z < (3 - \delta) = d$$
$$e = 1 - a - d$$

where Z is the standard normal variable.

Table 5: Power Functions for Location Using Detection Rules One and Four

shift δ	*k=1*	*k=2*	*k=3*	*k=4*	*k=5*	*k=6*	*k=7*	*k=8*	*k=9*	*k=10*
0	0.003	0.005	0.008	0.011	0.013	0.016	0.019	0.048	0.053	0.056
0.1	0.002	0.004	0.006	0.007	0.009	0.011	0.013	0.022	0.026	0.028
0.2	0.003	0.005	0.008	0.010	0.013	0.015	0.018	0.032	0.039	0.042
0.3	0.003	0.007	0.010	0.014	0.017	0.021	0.024	0.048	0.057	0.062
0.4	0.005	0.009	0.014	0.019	0.023	0.028	0.032	0.069	0.083	0.088
0.5	0.006	0.012	0.019	0.025	0.031	0.037	0.043	0.097	0.116	0.124
0.6	0.008	0.016	0.024	0.032	0.040	0.048	0.056	0.134	0.158	0.168
0.7	0.011	0.021	0.032	0.042	0.052	0.063	0.073	0.180	0.210	0.221
0.8	0.014	0.028	0.041	0.054	0.068	0.081	0.093	0.235	0.271	0.284
0.9	0.018	0.035	0.053	0.070	0.086	0.103	0.119	0.299	0.339	0.354
1.0	0.023	0.045	0.067	0.088	0.109	0.129	0.149	0.370	0.414	0.430
1.1	0.029	0.057	0.084	0.110	0.136	0.160	0.185	0.446	0.491	0.508
1.2	0.036	0.071	0.104	0.136	0.167	0.197	0.226	0.524	0.570	0.587
1.3	0.045	0.087	0.128	0.167	0.204	0.239	0.273	0.601	0.645	0.663
1.4	0.055	0.107	0.156	0.202	0.246	0.287	0.326	0.675	0.716	0.732
1.5	0.067	0.129	0.187	0.242	0.292	0.340	0.384	0.742	0.779	0.794
1.6	0.081	0.155	0.223	0.286	0.344	0.397	0.445	0.802	0.833	0.847
1.7	0.097	0.184	0.263	0.335	0.399	0.457	0.510	0.853	0.878	0.890
1.8	0.115	0.217	0.307	0.387	0.457	0.520	0.575	0.894	0.914	0.924
1.9	0.136	0.253	0.354	0.442	0.518	0.583	0.640	0.926	0.942	0.950
2.0	0.159	0.292	0.404	0.499	0.578	0.645	0.702	0.951	0.962	0.968
2.1	0.184	0.334	0.457	0.557	0.638	0.705	0.759	0.968	0.976	0.981
2.2	0.212	0.379	0.510	0.614	0.696	0.760	0.811	0.980	0.986	0.989
2.3	0.242	0.425	0.564	0.670	0.750	0.810	0.856	0.988	0.992	0.994
2.4	0.274	0.473	0.618	0.723	0.799	0.854	0.894	0.993	0.996	0.997
2.5	0.309	0.522	0.669	0.771	0.842	0.891	0.924	0.996	0.998	0.998
2.6	0.345	0.570	0.718	0.815	0.879	0.921	0.948	0.998	0.999	0.999
2.7	0.382	0.618	0.764	0.854	0.910	0.944	0.966	0.999	0.999	1.000
2.8	0.421	0.664	0.806	0.887	0.935	0.962	0.978	1.000	1.000	

(see Table 1 for larger shifts when k is less than 7)

Figure 10 compares the use of Detection Rules One and Four together with the use of Detection Rule One alone (when k = 10). In addition, the power function for the use of Detection Rules One and Two is also shown.

Bearing in mind that small differences in theoretical power are not likely to be appreciable in practice, we have to conclude that Rules One and Four will function essentially the same as Detection Rule One and Two.

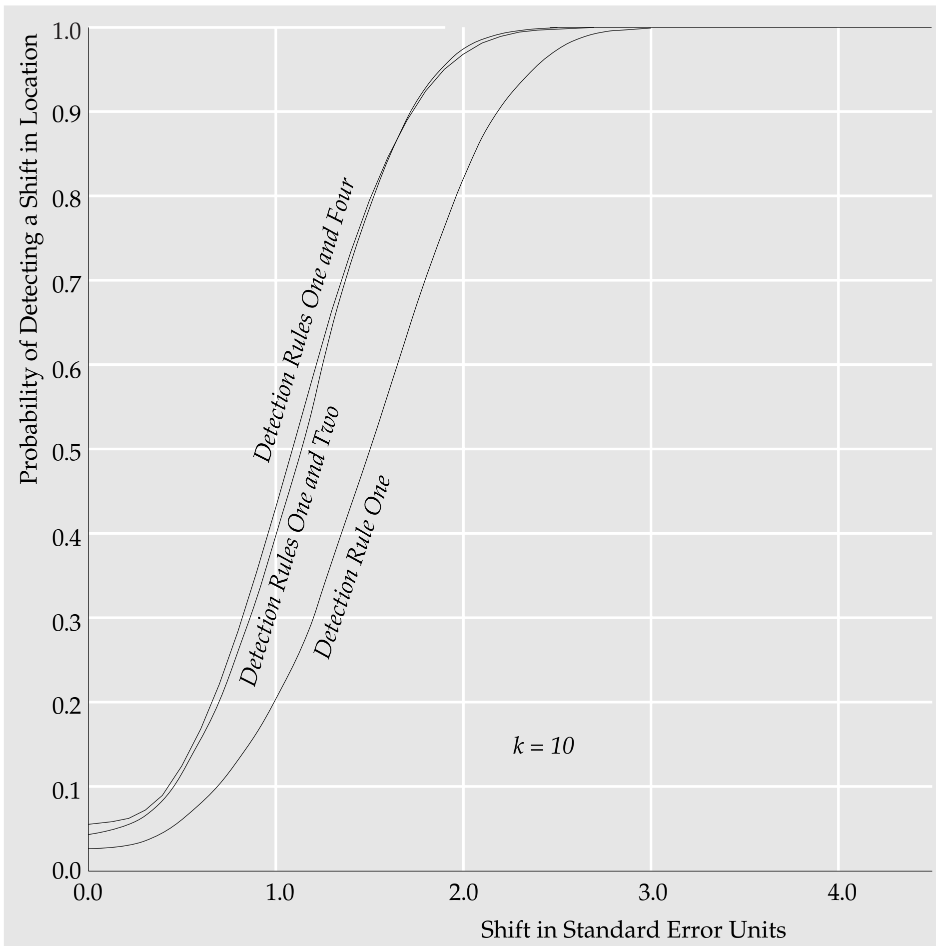

Figure 10: The Effect of Using Detection Rules One and Four

Once you are using Detection Rules One and Four (or Rules One and Two), the use of additional detection rules will only slightly increase the likelihood of detecting *very small shifts.* However, the use of more than Rules One and Four will always result in an increasing risk of a false alarm. Since the purpose of a process behavior chart is to detect those signals that are large enough to be of economic importance, it does not make sense to accept an increased risk of a false alarm in order to detect small signals that are unlikely to be of economic importance.

Nelson's Detection Rules One and Two

In 1984 Lloyd Nelson published a list of detection rules that incorporated Rules One, Two, and Three of the Western Electric Zone Tests. In place of Detection Rule Four Nelson substituted "nine points in a row" on the same side of the central line. Since many software programs use Nelson's list we include Nelson's alternative to using Western Electric Rules One and Four: his rules 1 and 2.

Here the probabilities of detecting a shift on the first eight subgroups following a shift are the same as for Rule One alone. The sum of these probabilities is neatly expressed by the well known formula:

$$\text{Probability of detecting shift within } k \text{ subgroups following shift } = \sum_{i=1}^{k} p_i$$

$$= \sum_{i=1}^{k} a\ (1-a)^{i-1}$$

$$= 1-(1-a)^{k}$$

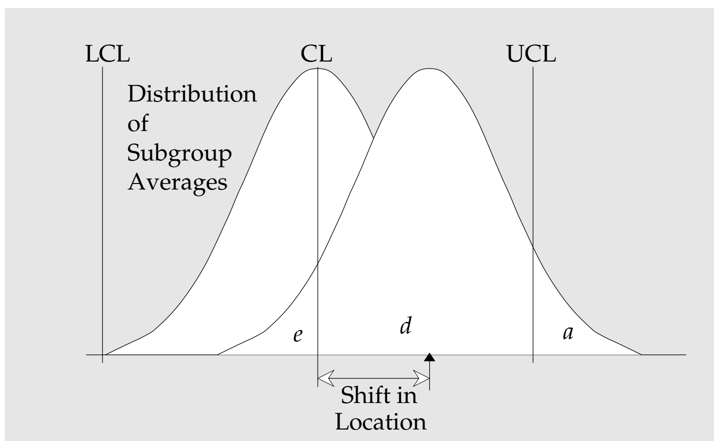

Figure 11: Probabilities for Rule One and "Nine in a Row"

Figure 11 shows the labels for the different probabilities associated with the use of Nelson's Rules One and Two. On the ninth and tenth subgroups there is a possibility of detecting a small shift with Nelson's Rule Two. The appropriate equations for p_9, and p_{10} are:

$$p_9 = a(1\text{-}a)^8 + d^9$$

$$p_{10} = 9\,ad^8e + 36\,ad^7e^2 + 84\,ad^6e^3 + 126\,ad^5e^4 + 126\,ad^4e^5 + 84\,ad^3e^6 + 36\,ad^2e^7 + 9\,ade^8e + ae^9 + d^9e$$

By summing these probabilities we can obtain the probability of detecting a shift within k subgroups following the shift using Nelson's Rules One and Two. These formulas are perfectly general—they will work with values of a, d, and e obtained from any appropriate probability distribution. Using the normal distribution, if we express the shift in standard error units as δ, then the probabilities needed for the formulas above will be:

$$\textit{Probability of } Z > (3 - \delta) \quad = a$$

$$\textit{Probability of } (0 - \delta) < Z < (3 - \delta) \quad = d$$

$$e = 1 - a - d$$

where Z is the standard normal variable.

Table 6: Power Functions for Location Using Nelson's Rules One and Two

shift δ	***k=1***	***k=2***	***k=3***	***k=4***	***k=5***	***k=6***	***k=7***	***k=8***	***k=9***	***k=10***
0	0.003	0.005	0.008	0.011	0.013	0.016	0.019	0.021	0.028	0.033
0.1	0.002	0.004	0.006	0.007	0.009	0.011	0.013	0.015	0.020	0.024
0.2	0.003	0.005	0.008	0.010	0.013	0.015	0.018	0.020	0.030	0.035
0.3	0.003	0.007	0.010	0.014	0.017	0.021	0.024	0.027	0.043	0.051
0.4	0.005	0.009	0.014	0.019	0.023	0.028	0.032	0.037	0.062	0.074
0.5	0.006	0.012	0.019	0.025	0.031	0.037	0.043	0.049	0.088	0.104
0.6	0.008	0.016	0.024	0.032	0.040	0.048	0.056	0.064	0.122	0.143
0.7	0.011	0.021	0.032	0.042	0.052	0.063	0.073	0.083	0.165	0.192
0.8	0.014	0.028	0.041	0.054	0.068	0.081	0.093	0.106	0.218	0.250
0.9	0.018	0.035	0.053	0.070	0.086	0.103	0.119	0.134	0.281	0.318
1.0	0.023	0.045	0.067	0.088	0.109	0.129	0.149	0.168	0.352	0.393
1.1	0.029	0.057	0.084	0.110	0.136	0.160	0.185	0.208	0.429	0.473
1.2	0.036	0.071	0.104	0.136	0.167	0.197	0.226	0.254	0.510	0.554
1.3	0.045	0.087	0.128	0.167	0.204	0.239	0.273	0.306	0.590	0.633
1.4	0.055	0.107	0.156	0.202	0.246	0.287	0.326	0.363	0.667	0.707
1.5	0.067	0.129	0.187	0.242	0.292	0.340	0.384	0.425	0.738	0.774
1.6	0.081	0.155	0.223	0.286	0.344	0.397	0.445	0.490	0.801	0.832
1.7	0.097	0.184	0.263	0.335	0.399	0.457	0.510	0.557	0.854	0.879
1.8	0.115	0.217	0.307	0.387	0.457	0.520	0.575	0.624	0.896	0.917
1.9	0.136	0.253	0.354	0.442	0.518	0.583	0.640	0.689	0.929	0.945
2.0	0.159	0.292	0.404	0.499	0.578	0.645	0.702	0.749	0.954	0.965
2.1	0.184	0.334	0.457	0.557	0.638	0.705	0.759	0.804	0.971	0.979
2.2	0.212	0.379	0.510	0.614	0.696	0.760	0.811	0.851	0.983	0.988
2.3	0.242	0.425	0.564	0.670	0.750	0.810	0.856	0.891	0.990	0.993
2.4	0.274	0.473	0.618	0.723	0.799	0.854	0.894	0.923	0.995	0.996
2.5	0.309	0.522	0.669	0.771	0.842	0.891	0.924	0.948	0.997	0.998
2.6	0.345	0.570	0.718	0.815	0.879	0.921	0.948	0.966	0.999	0.999
2.7	0.382	0.618	0.764	0.854	0.910	0.944	0.966	0.979	0.999	1.000
2.8	0.421	0.664	0.806	0.887	0.935	0.962	0.978	0.987	1.000	
2.9	0.460	0.709	0.843	0.915	0.954	0.975	0.987	0.993		
3.0	0.500	0.750	0.875	0.937	0.969	0.984	0.992	0.996		
3.1	0.540	0.788	0.903	0.955	0.979	0.991	0.996	0.998		
3.2	0.579	0.823	0.926	0.969	0.987	0.994	0.998	0.999		
3.3	0.618	0.854	0.944	0.979	0.992	0.997	0.999	1.000		

(see Table 1 for larger shifts when *k* is less than 8)

Figure 11 combines the last column of Tables 5 and 6 to show the effect of using Detection Rules One and Four together versus using Nelson's Rules One and Two for the case of k = 10. For comparison purposes, the power function for the use of Detection Rule One alone (from the last column of Table 1) is included.

Bearing in mind that the small differences in theoretical power seen in Figure 11 between the top two curves are not likely to be detectable in practice, we have to conclude that Nelson's Rules One and Two will function essentially the same as Western Electric's Detection Rules One and Four.

Nelson's rationale for using "nine in a row "instead of "eight in a row "was that it resulted in fewer false alarms. Here we see that in addition to having fewer false alarms, Nelson's Rule Two has essentially the same power to detect small and intermediate shifts as does Western Electric Rule Four.

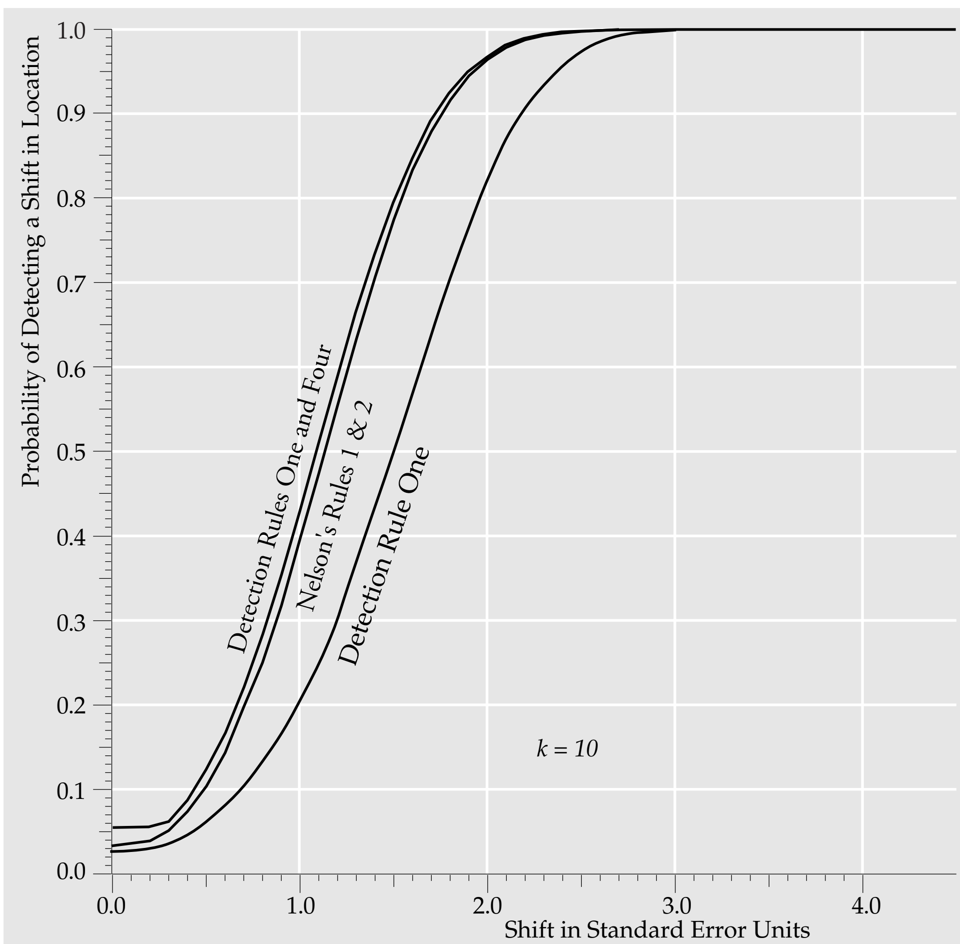

Figure 11: Detection Rules One and Four versus Nelson's Rules One and Two

Once you are using Nelson's Rules One and Two, it makes no sense to use additional detection rules. Adding additional detection rules will only slightly increase the likelihood of detecting *very small shifts*. However, the use of more than Nelson's Rules One and Two will always result in an increasing risk of a false alarm. Since the purpose of a process behavior chart is to detect those signals that are large enough to be of economic importance, it does not make sense to accept an increased risk of a false alarm in order to detect small signals that are unlikely to be of economic importance.

Summary

The use of the probability approach in this paper is a mathematical convenience, used to obtain answers to questions which cannot be answered in any other way. The theoretical values thus obtained will allow for direct comparisons to be made between different charting techniques, and between process behavior charts and other techniques. While such comparisons are useful, they are still strictly theoretical. Large differences in theoretical power may indicate a practical difference between the techniques, but small differences should not be expected to translate into detectable differences in practice.

Finally, you should understand that the theoretical probability model used to obtain the power functions in no way restricts the use of the process behavior charts, nor does it place prior restraints upon the application of process behavior charts. Three sigma limits, and run-tests are sufficiently conservative to work in the sequential analysis of a continuing stream of data regardless of what probability model might be thought to approximate the process output.